EDAA40 Exam

30 May 2017

Instructions

Things you CAN use during the exam.
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam.
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favorable assumption about what you intended to write.

A sheet with common symbols and notations is attached at the end.

Good luck!

Total points: 119

points required for 3: 60

points required for 4: 80

points required for 5: 101

1 [12 p]

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$. For any non-empty set S of numbers, $\max S$ and $\min S$ are the largest and smallest numbers S, respectively.

1. [3p] $\#\{(a, b) \in A \times A : a \leq b\} =$ ________
2. [4p] $\max\left\{\dfrac{a}{b} : a, b \in A\right\} - \min\left\{\dfrac{a}{b} : a, b \in A\right\} =$________
3. [5p] $\min\left\{\dfrac{a}{b} : a, b \in A, a > b\right\} - \max\left\{\dfrac{a}{b} : a, b \in A, a \leq b\right\} =$________

2 [20 p]

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ as before and a family of relations
$R_i = \{(a, b) \in A \times A : \mod (b, a) = i\}$ for any $i \in \mathbb{N}$, with $\mod (b, a)$ the remainder when dividing positive integer b by positive integer a. So, for example,
$R_3 = \{(a, b) \in A \times A : \mod (b, a) = 3\}$.

1. [2p] $\#R_4 =$
2. [2p] $\#R_5 =$
3. [2p] $\#R_0 =$
4. [2p] $\#R_{10} =$
5. [3p] $R_3(7) =$
6. [3p] $R_1(2) =$
7. [3p] $R_1(A) =$
8. [3p] $R_3(A) =$

   Hint: Please review the definition of “image”, and be sure of what the kind of result is expected here.

3 [16 p]

Suppose, as previously, $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ and a relation $R_3 = \{(a, b) \in A \times A : \mod (b, a) = 3\}$.

Let $T = R_3 \circ R_3^{-1}$.

1. [3 p] $\#T =$_______
2. [3 p] $T(1) =$____
3. [3 p] $T(7) =$____
4. [3 p] $T(A) =$____
5. [4 p] T is ... (circle those that apply)

… reflexive	TRUE	FALSE
… symmetric	TRUE	FALSE
… transitive	TRUE	FALSE
… antisymmetric	TRUE	FALSE

4 [18 p]

Suppose you have **injections** $f : A \hookrightarrow B$ and $g : A \hookrightarrow B$, as well as a **non-empty** set $S \subset A$ (note that S is a **proper** subset of A). Now let's define a function $h : A \longrightarrow B$ as follows:

$$h : x \mapsto \begin{cases} f(x) & \text{for } x \in S \\ g(x) & \text{for } x \notin S \end{cases}$$

This function is not, in general, injective.
Whether it is injective depends on the definitions of A, B, f, g, and S.

1. [5p] Give definitions for A, B, f, g, and S such that the h above is injective.

   A = ____

   B =____

   f =____

   g =____

   S =____

2. [5p] Give definitions for A, B, f, g, and S such that the h above is **not** injective.

   A = ____

   B =____

   f =____

   g =____

   S =____

3. [8p] Give a general formal criterion, depending only on A, B, f, g, and S (not necessarily all of them), that defines the condition under which h is injective. (Hint: Remember, f and g are already injective.)

   h is injective iff ____

Note: You are **not** supposed to reiterate the definition of injectivity for h, but rather give an expression involving at most A, B, f, g, and S (but **not** h) that is true if an only if they lead to an injective h.

5 [18 p]

Suppose we have a rooted tree (T, R) with nodes T, links $R \subseteq T \times T$, and root a as well as a labeling function $\lambda : T \longrightarrow \mathbb{N}$ assigning each node in the tree a natural number.

We want to define a function $L : T \longrightarrow \mathbb{N}$ that computes for each node $n \in T$ the lowest number a node in the subtree rooted at n is labeled with (that subtree includes n itself). If the subtree consists only of n, its label $\lambda(n)$ is the lowest number.

As before, for any non-empty set S of numbers, $\min S$ is the lowest number in that set.

1. [10p] Define L using well-founded recursion. (Hint: You may use cases if you like, but it is possible to define this function without an explicit "base case.")

$$L : n \mapsto$$

2. [8p] Define a strict partial order $\prec$ on T such that the poset $(T, \prec)$ is well-founded and your definition of L performs well-founded recursion on that poset. For all $n, n' \in T$...

$$n' \prec n \iff$$

No proof is required. It is sufficient that the strict partial order is well-founded and your definition of L conforms to it.

Hint: Make sure the partial order you define actually is one, i.e. that it has all the properties required from a strict partial order, including, for example, transitivity.

6 [10 p]

Suppose we have a set of four characters $C = \{"a", "b", "(", ")"\}$, the set $S = \{"a", "b"\}$ consisting only of the letters a and b, and a relation $R = \{(s_1, s_2, "(" s_1 "," s_2 ")") : s_1, s_2 \in C^*\}$.

As you can see, R is a 3-place relation, and we compute the image of some set of strings $X \subseteq C^*$ under R by applying the relation to all pairs of strings in X, i.e.
$R(X) = \{y : x_1, x_2 \in X, (x_1, x_2, y) \in R\}$.

Now let $R^n(X)$ be the set that results from computing the image of some set $X \subseteq C^*$ under R n times in a row, with $R^0(X) = X$, $R^1(X) = R(X)$, $R^2 = R(R(X))$ and so forth, so that $R^{n+1}(X) = R(R^n(X))$.

1. [1p] Give an element in $R^0(C)$:____________________

2. [2p] Give an element in $R^1(C)$:____________________

3. [2p] Give an element in $R^2(C)$:____________________

4. [5p] Suppose $U = \bigcup_{n \in \mathbb{N}} R^n(C)$.

   Give an element in $R[C] \setminus U$, or write "none" if no such element

   exists:____________________

   Note:
   $R[C]$ here is the closure of C under the set of relations that only consists of the relation R.

7 [10 p]

Identify free and bound occurrences of variables in the following formula. Put a dot **above** a free variable occurrence, and **below** a bound one.
Note that variable symbols immediately following quantifiers do not count as "occurrences".

free

$$((\forall z(Py \to Qzx)) \leftrightarrow Pz) \to (Qxz \wedge ((\exists x(Qxz)) \leftrightarrow \exists z(Pz \to Px)))$$

bound

8 [15 p]

Find a DNF for each of the following formulae. Write “none” if a formula has no DNF.

1. [5 p] $(r \vee \neg q) \leftrightarrow (p \wedge q)$

   ____

2. [5 p] $(p \barwedge (q \barwedge (r \barwedge s)))$

   ____

3. [5 p] $(p \to q) \barwedge ((q \to r) \barwedge (r \to p))$

   ____

Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e. including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k (k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- $\overline{R}$ of a relation R: its *complement*
- R^{-1} of a relation R: its *inverse*
- $R \circ S, f \circ g$ of relations and functions: their *composition*
- $R[X], f[X]$ *closure* of a set X under a relation R, a set of relations R, or a function f
- $[a, b],]a, b[,]a, b], [a, b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are *equinumerous*
- A^* for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, ε
- $\sum S$ sum of all elements of S
- $\prod S$ product of all elements of S
- $\bigcup S$ union of all elements in S
- $\bigcap S$ intersection of all elements in S
- $\bigcup\limits_{a \in S} E(s)$ generalized union of the sets computed for every s in S