EDAA40 Exam

28 May 2018

Instructions

Things you CAN use during the exam.
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam.
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favorable assumption about what you intended to write.

A sheet with common symbols and notations is attached at the end.

Good luck!

1	2	3	4	5	total
10	20	30	20	20	100

Total points: 100

points required for 3: 50

points required for 4: 67

points required for 5: 85

1 [10 p]

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ and a family of relations on A,

$$R_i = \{(a, b) \in A \times A : \mod (ab, a + b) = i\}$$

for any $i \in \mathbb{N}$. Here, $\mod (m, n)$ is the remainder when dividing positive integer m by positive integer n, e.g. $\mod (14, 4) = 2$, and $\mod (5, 7) = 5$.

For example, $R_3 = \{(a, b) \in A \times A : \mod (ab, a + b) = 3\}$.

Compute the following cardinalities:

1. [2 p] $\#R_5 =$
2. [2 p] $\#R_6 =$
3. [2 p] $\#R_9 =$
4. [4 p] $\#(R_5 \circ R_6) =$

2 [20 p]

Suppose a function $f : A \longrightarrow B$ and an injection $g : B \hookrightarrow C$. The sets A, B, and C are **not empty**, and they may or may not be the same set.

1. [5 p] Is their composition $g \circ f : A \longrightarrow C$ injective (circle answer)?

   Never Sometimes Always

   (Circle "Never" if a function f and an injective function g could never be composed in this way to result in an injective function. Circle "Sometimes", if some such compositions would be injective, others would not be, depending on f and g. Circle "Always" if all such compositions would be injective.)

2. [15 p] If you answered "Never" or "Always" to the question above, prove it. If you answered "Sometimes", construct an example where $g \circ f$ is injective and one where it is not. (Constructing an example would mean defining sets A, B, and C, as well as the two functions, according to the decription above.)

3 [30 p]

Suppose you have a graph (V, E) with vertices V and edges $E \subseteq V \times V$. Also given is a function $w : E \longrightarrow \mathbb{N}^+$, assigning each edge a positive natural number as its *edge weight*.

In this task, we represent a *path* in the graph as a finite sequence of elements in V. In other words: a path in this graph is an element of V^*, such that each pair of successive vertices in it are connected by an edge. For example, suppose $p = abdcfbe$. If $a, b, c, d, e, f \in V$ then $p \in V^*$, and if $(a, b), (b, d), (d, c), (c, f), (f, b), (b, e) \in E$, then p is a path in our graph. Its *path weight* is the sum of the edge weights of the edges in it. Note that the same vertex can appear any number of times in a path. The same is true for the edges, and if an edge occurs multiple times in a path, its edge weight is counted for each occurrence. For example, if $p = abdab$, then we add the edge weights for (a, b), (b, d), (d, a), and (a, b) again to get the path weight for p.

To keep things simple, we will consider ε, the empty sequence of nodes, a *path* for the purposes of this task.

We want to define a function

$$P : \mathbb{N} \longrightarrow \mathcal{P}(V^*)$$

which takes a natural number and computes a set of paths in the graph. Specifically, $P(n)$ is **the set of <u>all</u> paths** p in our graph such that

- p **visits every vertex in** V **at least once**, and
- its path weight is less than or equal to n.

In order to define P, we will use a helper function

$$P' : V \times V^* \times \mathcal{P}(V) \times \mathbb{N} \longrightarrow \mathcal{P}(V^*)$$

$P'(a, p, S, n)$ is the the set of all paths in our graph of the form paq, i.e. paths consisting of a path p, followed by vertex a, followed by path q, with the following properties:

- paq is a valid path in the graph,
- q visits all the vertices in S,
- the path weight of q is less than or equal to n.

Using P', we can define P as follows:

$$P : n \mapsto \bigcup_{a \in V} P'(a, \varepsilon, V \setminus \{a\}, n)$$

1. [25 p] Define P' recursively.

$P' : (a, p, S, n) \mapsto$

2. [5 p] In order to ensure that P' terminates, we require a **well-founded strict order** $\prec$ of its arguments, such that for any (a, p, S, n) that P' is called on, it will only ever call itself on $(a', p', S', n') \prec (a, p, S, n)$. Define such an order. (You are *not* asked to prove that it is well-founded, just to define it.)

$(a', p', S', n') \prec (a, p, S, n) \iff$

4 [20 p]

Suppose we have a tree (T, R) and a function $\lambda : T \longrightarrow \mathbb{N}$ assigning each node in the tree a natural number as a label.

1. [5 p] Define the set $T_{\mathbb{P}} \subseteq T$ consisting of all nodes that are labeled with a number in $\mathbb{P}$, i.e. a prime number (you do not have to define prime numbers, just use $\mathbb{P}$ in your definition):

   $T_{\mathbb{P}} =$

2. [5 p] Define the set $A_{x,y} \subseteq T$ consisting of all nodes that have label x and that have at least one child with label y (for $x, y \in \mathbb{N}$):

   $A_{x,y} =$

3. [5 p] Define the set $B_{x,y} \subseteq T$ consisting of all nodes that have label x and have at least one "grandchild", i.e. one child of a child, with label y (for $x, y \in \mathbb{N}$):

   $B_{x,y} =$

4. [5 p] Define the set $C_n \subseteq T$, consisting of all nodes that are descendents (children, children of children, children of children of children and so on) of node $n \in T$:

   $C_n =$

5 [20 p]

Find a DNF for each of the following formulae. Write "none" if a formula has no DNF.

1. [5 p] $((p \to q) \mathbin{\bar{\wedge}} (q \to r)) \wedge ((r \to s) \mathbin{\bar{\wedge}} (s \to p))$

2. [5 p] $\neg(((p \mathbin{\bar{\wedge}} q) \to (q \mathbin{\bar{\wedge}} r)) \to ((r \mathbin{\bar{\wedge}} s) \to (s \mathbin{\bar{\wedge}} p)))$

3. [5 p] $(p \leftrightarrow q) \wedge ((p \to r) \mathbin{\bar{\wedge}} (q \to s))$

4. [5 p] $(p \to q) \mathbin{\bar{\wedge}} ((p \leftrightarrow r) \mathbin{\bar{\wedge}} (q \leftrightarrow s))$

Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e. including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k (k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- $\overline{R}$ of a relation R: its *complement*
- R^{-1} of a relation R: its *inverse*
- $R \circ S, f \circ g$ of relations and functions: their *composition*
- $R[X], f[X]$ *closure* of a set X under a relation R, a set of relations R, or a function f
- $[a, b],]a, b[,]a, b], [a, b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are *equinumerous*
- A^* for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, ε
- $\sum S$ sum of all elements of S
- $\prod S$ product of all elements of S
- $\bigcup S$ union of all elements in S
- $\bigcap S$ intersection of all elements in S
- $\bigcup\limits_{a \in S} E(s)$ generalized union of the sets computed for every s in S