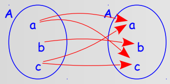
EDAA40
Discrete Structures in Computer Science



2: Relations

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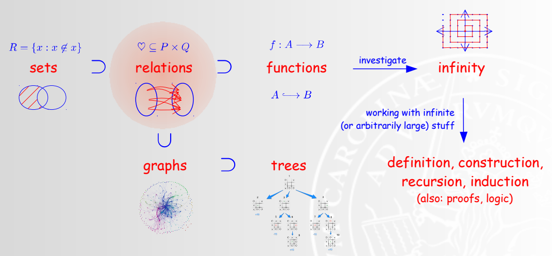

relations

Mathematical *relations* are about connections between objects.

relations between numbers
- a divides b, a is greater than b, a and b are prime to each other

relations between sets
- subset of, same size as, smaller than

relations between people
- customer/client, parent/child, spouse, employer/employee

We will focus on relations between two things. Often, they have distinct *roles* in a relation (superset/subset, parent/child, ...), i.e. we cannot model them simply as unordered pairs {a, b}.

In order to properly model relations, we first need to introduce *ordered pairs*.

ordered pairs, tuples

ordered pair (a, b)

$(a, b) = (x, y)$ iff $a = x$ and $b = y$

corollary:

$(a, b) \neq (b, a)$ if $a \neq b$

n-tuple $(a_1, ..., a_n)$

$(a_1, ..., a_n) = (b_1, ..., b_n)$ iff $a_i = b_i$ for $i = 1, ..., n$

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cartesian product

The *(cartesian) product* of a pair of sets, or more generally a finite family of sets, is the set of all ordered pairs or n-tuples.

$$A_1 \times ... \times A_n = \{(a_1, ..., a_n) : a_1 \in A_1, ..., a_n \in A_n\}$$

When the sets are the same, we also write

$A \times A = A^2$

$\underbrace{A \times ... \times A}_{n \text{ times}} = A^n$

If A and B are different, then

$A \times B \neq B \times A$

Occasionally, to avoid fussiness, the following are treated as equal:

$A \times (B \times C) = (A \times B) \times C = A \times B \times C$

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cartesian product

Examples:

$\{a, b\} \times \{1, 2, 3\} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

$\mathbb{N}^+ \times \mathbb{N}^+ = \{(1, 1), (1, 2), (1, 3), ..., (2, 1), (2, 2), (2, 3), ...\}$

Note: $\#(A \times B) = \#(A)\#(B)$

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relations

A (binary, dyadic) relation R from A to B
(*or over A x B*)
is a subset of the cartesian product: $R \subseteq A \times B$

If A and B are the same, i.e. $R \subseteq A \times A$, we also say that R is a binary relation *over A*.

Of course, this generalizes to...

An *n-place relation R over* $A_1 \times \ldots \times A_n$ is a subset of that product: $R \subseteq A_1 \times \ldots \times A_n$

notation, examples

For binary relations $R \subseteq A \times B$, these are equivalent:

$(a, b) \in R$ $\qquad$ aRb

$C = \{F, E, B, D, NL, CH, I, GB, IRL\}$

$\bowtie \; = \{(F,E),(E,F),(F,B),(B,F),(F,D),(D,F),$
$(F,CH),(CH,F),(F,I),(I,F),(B,NL),(NL,B),$
$(B,D),(D,B),(D,NL),(NL,D),(D,CH),(CH,D),$
$(CH,I),(I,CH),(GB,IRL),(IRL,GB)\}$

Therefore: $F \bowtie CH$ but $E \not\bowtie I$

examples

$< \; \subseteq \mathbb{N}^+ \times \mathbb{N}^+$
$< \; = \{(1,2),(1,3),...,(1,1557),...,(2,3),(2,4),....\}$
$(4,7) \in <$ but $(2,2) \notin <$ and $(7,1) \notin <$

Suppose $\{M_i : i \in \mathbb{N}\}$ with $M_i = \{ik : k \in \mathbb{N}^+\}$

Let's define the relation

$| = \{(a,b) \in \mathbb{N}^+ \times \mathbb{N}^+ : b \in M_a\}$

? What does this relation signify?
When is $a \mid b$?

terminology: source, target, domain, range

For binary relations $R \subseteq A \times B$:
A is a *source*. B is a *target*.

Note that for any R, source and target are *not uniquely determined*

$$R \subseteq A \times B$$

For any $A' \supseteq A$ and $B' \supseteq B$, we have $A \times B \subseteq A' \times B'$.

$$R \subseteq A \times B \subseteq A' \times B'$$

By contrast, these *are* uniquely determined:
the *domain* of R: $\mathrm{dom}(R) = \{a : (a,b) \in R \text{ for some } b\}$
the *range* of R: $\mathrm{range}(R) = \{b : (a,b) \in R \text{ for some } a\}$

For any relation $R \subseteq A \times B$ it is always the case that
$\mathrm{dom}(R) \subseteq A$ and $\mathrm{range}(R) \subseteq B$

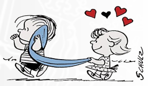
example

$R_{\text{Charlie}} = \{\text{Violet}, \text{LRHG}, \text{Peggy}\}, R_{\text{Linus}} = \{\text{Sally}, \text{Mrs. Othmar}, \text{Lydia}\}, R_{\text{Lucy}} = \{\text{Schroeder}\},$
$R_{\text{Patty}} = \{\text{Charlie}\}, R_{\text{Sally}} = \{\text{Linus}\}, R_{\text{Violet}} = \{\text{Violet}\}, R_{\text{Peggy}} = \{\text{Charly}\}$
$P = \{\text{Charlie}, \text{Linus}, \text{Lucy}, \text{Patty}, \text{Sally}, \text{Violet}, \text{Peggy}, \text{Lydia}, \text{Schroeder}\}$
$Q = \{\text{Charlie}, \text{Linus}, \text{Lucy}, \text{Patty}, \text{Sally}, \text{Violet}, \text{Peggy}, \text{Lydia}, \text{Schroeder}, \text{LRHG}, \text{Mrs. Othmar}\}$

We can represent the same information as a relation from P to Q:

$\heartsuit \subseteq P \times Q$

$\heartsuit = \{(\text{Charlie}, \text{Violet}), (\text{Charlie}, \text{LRHG}), (\text{Charlie}, \text{Peggy}),$
$(\text{Linus}, \text{Sally}), (\text{Linus}, \text{Mrs. Othmar}), (\text{Linus}, \text{Lydia}),$
$(\text{Lucy}, \text{Schroeder}), (\text{Patty}, \text{Charlie}), (\text{Sally}, \text{Linus}),$
$(\text{Violet}, \text{Violet}), (\text{Peggy}, \text{Charlie})\}$



So that $\text{Sally} \heartsuit \text{Linus}$ but $\text{Sally} \not\heartsuit \text{Schroeder}$.

relations as tables

♡	Charlie	Linus	Lucy	Patty	Sally	Violet	Peggy	Lydia	Schroeder	LRHG	Mrs Othmar
Charlie	0	0	0	0	0	1	1	0	0	1	0
Linus	0	0	0	0	1	0	0	1	0	0	1
Lucy	0	0	0	0	0	0	0	0	1	0	0
Patty	1	0	0	0	0	0	0	0	0	0	0
Sally	0	1	0	0	0	0	0	0	0	0	0
Violet	0	0	0	0	0	1	0	0	0	0	0
Peggy	1	0	0	0	0	0	0	0	0	0	0
Lydia	0	0	0	0	0	0	0	0	0	0	0
Schroeder	0	0	0	0	0	0	0	0	0	0	0

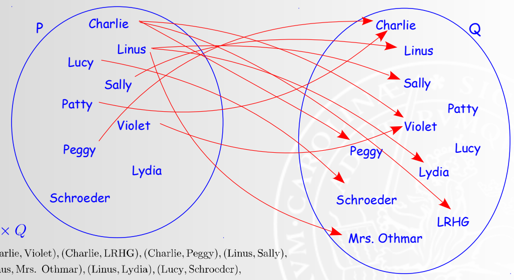
Q

P

$\heartsuit \subseteq P \times Q$

$\heartsuit = \{(\text{Charlie}, \text{Violet}), (\text{Charlie}, \text{LRHG}), (\text{Charlie}, \text{Peggy}), (\text{Linus}, \text{Sally}),$
$(\text{Linus}, \text{Mrs. Othmar}), (\text{Linus}, \text{Lydia}), (\text{Lucy}, \text{Schroeder}),$
$(\text{Patty}, \text{Charlie}), (\text{Sally}, \text{Linus}), (\text{Violet}, \text{Violet}), (\text{Peggy}, \text{Charlie})\}$

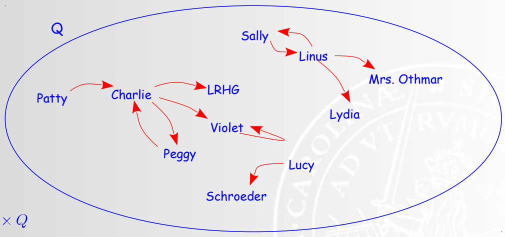
drawing relations: digraphs



$\heartsuit \subseteq P \times Q$

$\heartsuit = \{(\text{Charlie}, \text{Violet}), (\text{Charlie}, \text{LRHG}), (\text{Charlie}, \text{Peggy}), (\text{Linus}, \text{Sally}),$
$(\text{Linus}, \text{Mrs. Othmar}), (\text{Linus}, \text{Lydia}), (\text{Lucy}, \text{Schroeder}),$
$(\text{Patty}, \text{Charlie}), (\text{Sally}, \text{Linus}), (\text{Violet}, \text{Violet}), (\text{Peggy}, \text{Charlie})\}$

drawing relations: digraphs



$\heartsuit \subseteq Q \times Q$

$\heartsuit = \{(\text{Charlie}, \text{Violet}), (\text{Charlie}, \text{LRHG}), (\text{Charlie}, \text{Peggy}), (\text{Linus}, \text{Sally}),$
$(\text{Linus}, \text{Mrs. Othmar}), (\text{Linus}, \text{Lydia}), (\text{Lucy}, \text{Schroeder}),$
$(\text{Patty}, \text{Charlie}), (\text{Sally}, \text{Linus}), (\text{Violet}, \text{Violet}), (\text{Peggy}, \text{Charlie})\}$

converse, complement

For a binary relation $R \subseteq A \times B$
its *converse* (*inverse*) is the relation $R^{-1} = \{(b, a) : aRb\}$

some properties:
$R^{-1} \subseteq B \times A$
$(R^{-1})^{-1} = R$
$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$ $\quad (R \cap S)^{-1} = R^{-1} \cap S^{-1}$

For a binary relation $R \subseteq A \times B$
its *complement* is the relation $\overline{R} = -_{A \times B} R = A \times B \setminus R$

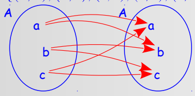
some properties:
$\overline{R} \subseteq A \times B$ $\quad \overline{\overline{R}} = R$
$\overline{R \cup S} = \overline{R} \cap \overline{S}$ $\quad \overline{R \cap S} = \overline{R} \cup \overline{S}$

Notation: There is no firm standard for denoting converse or complement. When using symbols such as $\prec$ or $\bowtie$, the complement is often indicated by striking through the symbol, i.e. $\not\prec$ or $\not\bowtie$, while the converse is denoted by reversing the symbol $\succ$.

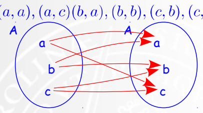
converse vs complement

Especially when source and target are the same, converse and complement seem to have a lot in common. Hence the importance of understanding the differences.

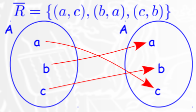
$R = \{(a,a),(a,b),(b,b),(b,c),(c,a),(c,c)\}$



$R^{-1} = \{(a,a),(a,c)(b,a),(b,b),(c,b),(c,c)\}$



$\overline{R} = \{(a,c),(b,a),(c,b)\}$



converse: invert the arrows
complement: absent arrows

For finite A, B, given $R \subseteq A \times B$
What are $\#(R^{-1})$ and $\#(\overline{R})$?

converse vs complement

$R = \{(a,a),(a,b),(b,b),(b,c),(c,a),(c,c)\}$

R	a	b	c
a	1	1	0
b	0	1	1
c	1	0	1

$R^{-1} = \{(a,a),(a,c)(b,a),(b,b),(c,b),(c,c)\}$

R	a	b	c
a	1	0	1
b	1	1	0
c	0	1	1

converse: mirror at the diagonal

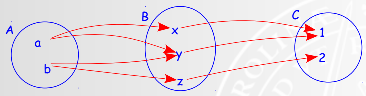
$\overline{R} = \{(a,c),(b,a),(c,b)\}$

R	a	b	c
a	0	0	1
b	1	0	0
c	0	1	0

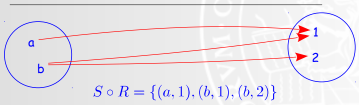
complement: flip zeros and ones

composition

Given two binary relations $R \subseteq A \times B$ and $S \subseteq B \times C$
their *composition* is a binary relation on $A \times C$
$S \circ R = \{(a,c) : aRb \text{ and } bSc \text{ for some } b \in B\}$



$R = \{(a,x),(a,y),(b,y),(b,z)\}$ $S = \{(x,1),(y,1),(z,2)\}$



$S \circ R = \{(a,1),(b,1),(b,2)\}$

composition

sidebar

$R = \{(a,x),(a,y),(b,y),(b,z)\}$

R	x	y	z
a	1	1	0
b	0	1	1

$S = \{(x,1),(y,1),(z,2)\}$

S	1	2
x	1	0
y	1	0
z	0	1

$S \circ R = \{(a,1),(b,1),(b,2)\}$

SoR	1	2
a	1	0
b	1	1

What is the relationship between the tables for R and S, and their composition?

image

Given a binary relation $R \subseteq A \times B$ from A to B, for any $a \in A$ its *image under R*, written R(a), is defined as $R(a) = \{b \in B : aRb\}$

Can be "lifted" to subsets $X \subseteq A$: $R(X) = \{b \in B : aRb \text{ for some } a \in X\}$

Note: $R(X) = \bigcup_{a \in X} R(a)$

$C = \{F, E, B, D, NL, CH, I, GB, IRL\}$

$\bowtie \; = \{(F,E),(E,F),(F,B),(B,F),(F,D),(D,F),(F,CH),(CH,F),(F,I),$
$(I,F),(B,NL),(NL,B),(B,D),(D,B),(D,NL),(NL,D),(D,CH),$
$(CH,D),(CH,I),(I,CH),(GB,IRL),(IRL,GB)\}$

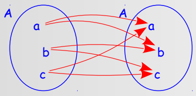
1. What is $\bowtie(F)$?
2. What does it mean?

properties: reflexivity

A binary relation $R \subseteq A \times A$ is *reflexive* iff for all $a \in A$ aRa

A binary relation $R \subseteq A \times A$ is *irreflexive* iff there is no $a \in A$ such that aRa

$R = \{(a,a),(a,b),(b,b),(b,c),(c,a),(c,c)\}$

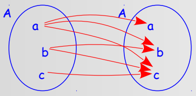


R	a	b	c
a	1	1	0
b	0	1	1
c	1	0	1

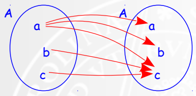
Other examples?
What is the difference between irreflexive and not reflexive?

properties: transitivity

A binary relation $R \subseteq A \times A$ is *transitive* iff for all $a, b, c \in A$
if aRb and bRc then aRc



R	a	b	c
a	1	1	1
b	0	1	1
c	0	0	1

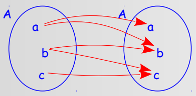


R	a	b	c
a	1	1	1
b	0	0	1
c	0	0	1

Other examples?

properties: transitivity (postscriptum)

A binary relation $R \subseteq A \times A$ is *transitive* iff for all $a, b, c \in A$
if aRb and bRc then aRc



R	a	b	c
a	1	1	0
b	0	1	1
c	0	0	1

In the lecture, I messed up the presentation of the previous slide, by suggesting that the second example on it was a counterexample that wasn't transitive, when in fact it was (transitive).

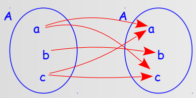
Here is an actual counterexample that isn't transitive. Promise.

aRb and bRc, but not aRc

properties: symmetry

A binary relation $R \subseteq A \times A$ is *symmetric* iff for all $a, b \in A$
if aRb then bRa

$R = \{(a,a), (a,c), (b,b), (c,a), (c,c)\}$



R	a	b	c
a	1	0	1
b	0	1	0
c	1	0	1

Other examples?

properties: a(nti)symmetry

Consider $\leq$ and $<$ on the natural numbers. Neither is symmetric, but in slightly different ways.
For $<$, it is **never** the case that $a < b$ and $b < a$.
This is called **asymmetry**.

For $\leq$, it sometimes is, but only when $a = b$.
This is called **antisymmetry**.

Both relations are antisymmetric. Only $<$ is asymmetric.

A binary relation $R \subseteq A \times A$ is *asymmetric* iff for all $a, b \in A$
if aRb then not bRa

A binary relation $R \subseteq A \times A$ is *antisymmetric* iff for all $a, b \in A$
if aRb and bRa then $a = b$

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equivalence relations

A binary relation $\approx \subseteq A \times A$ is an *equivalence relation* iff it is
1. reflexive
2. symmetric
3. transitive

What about these:
- equality
- having the same number of elements: $A \sim B$ iff $\#(A) = \#(B)$
- divides: $m \mid n$ iff there is $k \geq 1 : km = n$
- relatively prime: $m \perp n$ iff there is no $k \geq 2 : k \mid m$ and $k \mid n$

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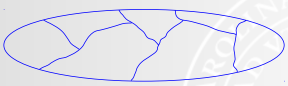
partitions

Given a set A, a *partition of A* is a set of pairwise disjoint sets $\{B_i : i \in I\}$, such that $A = \bigcup_{i \in I} B_i$



A: EU citizens, I: EU member states, B_i: citizens of country i
A: atoms, I: elements, B_i: atoms of element i
A: natural numbers, I: primes, B_i: multiples of i (excluding i)

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equivalence class, quotient set

Equivalence relations and partitions are really the same thing!

Given a set A and an equivalence relation $\approx$ on A, for any $a \in A$ we define the *equivalence class of a* $[a]_\approx$ as $[a]_\approx = \{b \in A : a \approx b\}$

Alternative syntax: $[a]$, $|a|$ when the relation is understood; SLAM $\left\{ \begin{array}{l} |a| \\ |a|_\approx \end{array} \right.$

Given a set A and an equivalence relation $\approx$ on A, the *quotient (set)* $A/\approx$ is defined as $A/\approx \; = \{|a|_\approx : a \in A\}$

SLAM 2.5.4:
1. Every partition is the quotient of an equivalence relation.
2. Every quotient set is a partition.

Review the proof in the book. Connect it to these definitions.

order relation, poset

A binary relation $\preceq \subseteq A \times A$ is an (*inclusive* or *non-strict*) (*partial*) *order* iff it is
1. reflexive
2. antisymmetric
3. transitive

What about these:
- divides: $m \mid n$ iff there is $k \geq 1 : km = n$
- set inclusion: $\subseteq$
- on numbers: $\leq$ and $<$
- proper set inclusion: $\subset$

A pair $(A, \preceq)$ where A is a set and $\preceq \subseteq A \times A$ a partial order on A is called a *partially ordered set* or *poset*.

Examples: $(\mathbb{N}^+, |)$
$(\mathcal{P}(A), \subseteq)$

strict (partial) order

A binary relation $\prec \subseteq A \times A$ is a *strict* (*partial*) *order* iff it is
1. irreflexive
2. transitive

Note: Irreflexivity and transitivity imply asymmetry.

How?

irreflexivity: $a \not\prec a$
transitivity: if $a \prec b$ and $b \prec c$ then $a \prec c$
asymmetry: if $a \prec b$ then $b \not\prec a$

total (or linear) order

A binary relation $\preceq \subseteq A \times A$ is a (*non-strict*) *total (or linear) order* iff it is
1. reflexive
2. antisymmetric
3. transitive
4. total (complete): $a \preceq b$ or $b \preceq a$

What about these:
- divides: $m \mid n$ iff there is $k \geq 1 : km = n$
- set inclusion: $\subseteq$
- on numbers: $\leq$ and $<$

transitive closure

The *transitive closure* R^+ of a binary relation $R \subseteq A \times A$ is defined as follows:

$$R^+ = \bigcup_{i \in \mathbb{N}} R_i \text{ with}$$
$$R_0 = R$$
$$R_{n+1} = R_n \cup \{(a, c) : \text{if } aR_nb \text{ and } bR_nc \text{ for some } b \in A\}$$

R^*
alternative syntax (SLAM)

$$\bowtie \; = \{(F, E), (E, F), (F, B), (B, F), (F, D), (D, F), (F, CH), (CH, F), (F, I), (I, F), (B, NL), (NL, B), (B, D), (D, B), (D, NL), (NL, D), (D, CH), (CH, D), (CH, I), (I, CH), (GB, IRL), (IRL, GB)\}$$

What is the meaning of $\bowtie^+$?
What are its properties?