EDAA40
Discrete Structures in Computer Science

∀ 8: Quantificational logic ∃

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objective

You should be able to read, understand, and write quantificational logic.

logic with quantifiers (informally)

quantifying *over a set*

In practice, we are usually interested in speaking about elements of some set.
In such cases, the set is often specified when the variable is introduced:

$\forall x \in A\ (\alpha)$ represents "*for all x in A, α*"

$\exists x \in A\ (\alpha)$ represents "*there exists an x in A, such that α*"

This is just syntactic sugar:

$\forall x \in A\ (\alpha) \dashv\vdash \forall x (x \in A \to \alpha)$

$\exists x \in A\ (\alpha) \dashv\vdash \exists x (x \in A \land \alpha)$

True or false?

$\forall r \in \mathbb{R}\ (\exists n \in \mathbb{N}\ (n = r))$

$\exists n \in \mathbb{N}\ (\forall r \in \mathbb{R}\ (n = r))$

$\forall n \in \mathbb{N}\ (\exists r \in \mathbb{R}\ (n = r))$

$k|n \leftrightarrow \ldots\ \exists a \in \mathbb{Z}\ (ak = n)$

$n \in \mathbb{P} \leftrightarrow \ldots n \in \mathbb{N}_2 \land \forall k \in \mathbb{N}_2\ (k|n \to k = n)$

$\mathbb{N}_2 = \{n \in \mathbb{N} : n > 1\}$

more syntactic sugar

Often, one quantifier is used to introduce several variables:

$\forall x, y, z \in A\ (\alpha)$ represents "*forall x, y, z in A, α*"

$\exists x, y, z \in A\ (\alpha)$ represents "*there exist x, y, z in A, such that α*"

This, too, is just syntactic sugar:

$\forall x, y, z \in A\ (\alpha) \dashv\vdash \forall x \in A\ (\forall y \in A\ (\forall z \in A\ (\alpha)))$

$\exists x, y, z \in A\ (\alpha) \dashv\vdash \exists x \in A\ (\exists y \in A\ (\exists z \in A\ (\alpha)))$

True or false?

$\forall n \in \mathbb{N}\ (\exists a, b \in \mathbb{N}\ (a < n < b))$

$\forall a, b \in \mathbb{N}\ (\exists n \in \mathbb{N}\ (a < n < b))$

$\forall a, b \in \mathbb{N}\ (a < b \to \exists n \in \mathbb{N}\ (a < n < b))$

How can we "fix" this?

$R \subseteq A \times A$ transitive $\leftrightarrow \ldots$

$\forall a, b, c \in A\ (aRb \land bRc \to aRc)$

$\forall a, b \in A\ (aRb \to R(b) \subseteq R(a))$

$\forall (a, b) \in R\ (R(b) \subseteq R(a))$

the language of quantificational logic

more examples...

Zermelo-Fraenkel Set Theory w/Choice (ZFC)

- extensionality $\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$.
- regularity $\forall x [\exists a (a \in x) \Rightarrow \exists y (y \in x \wedge \neg \exists z (z \in y \wedge z \in x))]$.
- specification $\forall w_1, \ldots, w_n \forall A \exists B \forall x (x \in B \Leftrightarrow [x \in A \wedge \varphi(x, w_1, \ldots, w_n, A)])$
- union $\forall \mathcal{F} \exists A \forall Y \forall x [(x \in Y \wedge Y \in \mathcal{F}) \Rightarrow x \in A]$.
- replacement $\forall A \forall w_1 \forall w_2 \ldots \forall w_n [\forall x (x \in A \Rightarrow \exists! y \, \phi) \Rightarrow \exists B \, \forall x (x \in A \Rightarrow \exists y (y \in B \wedge \phi))]$.
- infinity $\exists X [\varnothing \in X \wedge \forall y (y \in X \Rightarrow S(y) \in X)]$.
- power set $\forall x \exists y \forall z [z \subseteq x \Rightarrow z \in y]$.
- choice $\forall X \left[\emptyset \notin X \implies \exists f : X \to \bigcup X \quad \forall A \in X \, (f(A) \in A)\right]$.

$S(x) = x \cup \{x\}$

finite transforms

Suppose we quantify all variables over a finite set D, and we have constant symbols $a_1, \ldots, a_n$ for each of its elements.

A *finite transform* of a universally/existentially quantified formula removes the quantifier, and instantiates the body for each element of D in a chained conjunction/disjunction.

Example: $\forall x(Px)$ becomes $Pa_1 \wedge \ldots \wedge Pa_n$

$\exists x(Px)$ becomes $Pa_1 \vee \ldots \vee Pa_n$

Do this for the following formula, until all quantifiers are gone. $\exists x(Qx \to \forall y(Pxy))$
Assume a domain with two values, with constant names a and b.

$\exists x(Qx \to (Pxa \wedge Pxb))$ $(Qa \to (Paa \wedge Pab)) \vee (Qb \to (Pba \wedge Pbb))$

equivalences: distribution

equivalences: quantifier interchange

The following equivalences hold for any formula α :

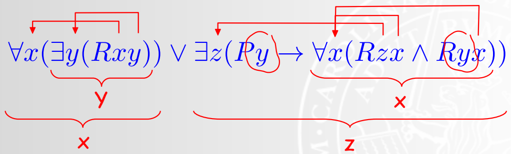
$$\forall x(\alpha) \dashv\vdash \neg\exists x(\neg\alpha)$$
$$\exists x(\alpha) \dashv\vdash \neg\forall x(\neg\alpha)$$

Remember de Morgan's laws?

$$\alpha \wedge \beta \dashv\vdash \neg(\neg\alpha \vee \neg\beta)$$
$$\alpha \vee \beta \dashv\vdash \neg(\neg\alpha \wedge \neg\beta)$$

quantifier scopes

variable *uses*, and the quantifier they are bound by



quantifier scopes, and the variables bound in/by them

free and bound variable occurrences

equivalences: vacuity, relettering

Vacuity: If x does not occur free in α , then

$$\forall x(\alpha) \dashv\vdash \alpha \dashv\vdash \exists x(\alpha)$$

This doesn't work if quantifying over an empty set.

Relettering: If x does not occur at all in α , and α' is the result of replacing every bound occurrence of some variable y in α with x, then

$$\alpha \dashv\vdash \alpha'$$

Example:

$$\forall y(Ry \to Qyz) \dashv\vdash \forall x(Rx \to (Qxz))$$

interpretations

The value of a formula depends on how you read the symbols in it.

$\forall z(Rxz \to \exists y(Rzy))$ $\quad \forall k \in \mathbb{N}_2\ (k|n \to k = n)$ $\quad \exists a \in \mathbb{Z}\ (ak = n)$

Also, we need to determine what values the quantified variables can assume.

A *domain* or *universe* D are the values that quantified variables range over. (For example: all sets in the case of the axioms of set theory.)

An *interpretation* v is a function assigning mathematical objects to the symbols occurring in a formula. Specifically...

- to each constant a $\quad v(a) \in D$
- to each variable x $\quad v(x) \in D$
- to each n-place function letter f $\quad v(f) : D^n \longrightarrow D$
- to each n-place relation letter P $\quad v(P) \subseteq D^n$
- to the identity symbol $\equiv$ the identity over D

evaluating terms and formulae

logical implication

Given a set of formulae $A = \{\alpha_1, ..., a_n\}$ and a formula β, we say that A *logically implies* β iff there is no interpretation v such that all the formulae in A are true under v, but β is false:

$$A \vdash \beta \iff \neg \exists v (\neg v(\beta) \wedge \forall \alpha (\alpha \in A \rightarrow v(\alpha)))$$

Also:

$\alpha \dashv\vdash \beta \iff \alpha \vdash \beta \wedge \beta \vdash \alpha$ logical equivalence

$\emptyset \vdash \alpha$ logical truth

$\emptyset \vdash \neg\alpha$ contradiction

A formula that is neither logically true nor a contradiction is *contingent*.

some implications