Lecture 5: In-class exercises

Direct Proof

1.

Show that for any integer x, if x is even, then x^2 is even.

Since x is even, there exists an integer a such that x = 2a. Therefore, $x^2 = 4a^2 = 2(2a^2)$. Since $2a^2$ is an integer, x^2 is even.

2.

For any two integers a, b, we say that a divides b, and write a|b, iff there is an integer k such that ak = b.

Show that for any three integers a, b, c, if a|b and b|c, then a|c.

Since a|b, b|c there exist integers k_1, k_2 such that $ak_1 = b$ and $bk_2 = c$. Substituting the first equation into the second, we get $ak_1k_2 = c$, and since k_1k_2 is an integer, a|c.

3.

Show that for any two injections $f : A \longrightarrow B$ and $g : B \longrightarrow C$, their composition $g \circ f : A \longrightarrow C$ is injective.

To prove their composition injective, we need to show that for any $a_1, a_2 \in A$ s.t. $a_1 \neq a_2$, it is the case that $g \circ f(a_1) \neq g \circ f(a_2)$.

Since *f* is injective, we get $f(a_1) \neq f(a_2)$. Since *g* is injective, we get $g(f(a_1)) \neq g(f(a_2))$. By definition of composition, this means $g \circ f(a_1) \neq g \circ f(a_2)$.

Direct Proof with Cases

4.

Show that every multiple of 4 equals $1 + (-1)^n (2n-1)$ for some $n \in \mathbb{N}$.

Hint: If *k* is a multiple of 4, it means there is an integer $a \in \mathbb{Z}$ such that k = 4a. For this proof, it helps to use the cases a = 0, a > 0, and a < 0.

[cf. BoP, p. 99]

Contrapositive Proof

5.

Show that for any integers $x, y \in \mathbb{Z}$, if 5 does not divide xy then 5 does not divide x, and it also does not divide y.

Hint 1: The logical negation of (A and B) is (not A **or** not B).

Hint 2: This proof is best done using cases.

[cf. BoP, p. 105]

Proof by Contradiction

6.

Show that the number $\sqrt{2}$ is irrational.

Hint 1: The opposite of a number being irrational is that it can be represented as a fraction $\frac{a}{b}$ of integers *a*, *b*. It is useful to require that the fraction be fully reduced (that's how we will produce the contradiction in this case), i.e. the two integers do not have a common divisor.

Hint 2: In particular, they cannot both be even, because that would mean that 2 is a common divisor.

[cf. BoP, p. 113]