Exercises 1 — sets

1.

Using set builder notation, define the following sets (you can use the usual sets of numbers as starting points, see the last page):

1. The set E of all even natural numbers (starting at 0).

   $E =$

2. Given the set $D = \{d \in \mathbb{N} : d \leq 9\}$, define the family of sets $\{L_d : d \in D\}$ such that L_d is the set of all natural numbers whose decimal representation ends with the digit d.

   $L_d =$

3. The set T of natural numbers that are the product of exactly three different primes.

   $T =$

4. The set X of pairs of natural numbers such that in each pair, the left-hand number is a square number, and the right-hand number is the cube of the same value that the left-hand number is the square of.

   $X =$

2.

Given $A = \{a, b, c, d, e, f\}$, what is

1. $\#(\{s \in \mathcal{P}(A) : \#(s) = 0\}) =$
2. $\#(\{s \in \mathcal{P}(A) : \#(s) = 1\}) =$
3. $\#(\{s \in \mathcal{P}(A) : \#(s) = 2\}) =$
4. $\#(\{s \in \mathcal{P}(A) : \#(s) = 3\}) =$
5. $\#(\{s \in \mathcal{P}(A) : \#(s) = 4\}) =$
6. $\#(\{s \in \mathcal{P}(A) : \#(s) = 5\}) =$

7. $\#(\{s \in \mathcal{P}(A) : \#(s) = 6\}) =$

3.

Given $A = \{3, 4, 5, 6, 7, 8\}$, suppose we define the following sets

$$B = \left\{ \frac{a-b}{a+b} : a, b \in A \right\}$$

$$C = \left\{ \frac{a}{b} : a, b \in A \wedge a \geq b \right\}$$

Give the number of elements in these sets as follows:

1. $\#(B) =$______
2. $\#(C) =$______

4.

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$. For any non-empty set S of numbers, $\max S$ and $\min S$ are the largest and smallest numbers S, respectively.

1. $\#\{(a, b) \in A \times A : a \leq b\} =$
2. $\max \left\{ \frac{a}{b} : a, b \in A \right\} - \min \left\{ \frac{a}{b} : a, b \in A \right\} =$
3. $\min \left\{ \frac{a}{b} : a, b \in A, a > b \right\} - \max \left\{ \frac{a}{b} : a, b \in A, a \leq b \right\} =$

5.

Given $A = \{a \in \mathbb{N}^+ : a \leq 6\}$, let us define the following sets:

$$B = \left\{ \frac{a}{b} : a, b \in A \right\}$$
$$C = \left\{ \frac{a}{b} : a, b \in A, b|a \right\}$$
$$D = \left\{ \frac{a}{b} : a, b \in A, a|b \right\}$$

Give the number of elements in these sets as follows:

1. $\#(B) =$
2. $\#(C) =$
3. $\#(D) =$

6.

Given $A = \{1, 2, 3, 4, 5, 6, 7\}, B = \{3, 4, 5, 6, 7, 8\},\ C = \{ab : a \in A, b \in B\}$
$D = \{ab : a \in A, b \in B, a > b\}$, and $E = \{ab : a \in A, b \in A \cap B\}$:

1. $\#(C) =$
2. $\#(D) =$
3. $\#(E) =$

7.

Given $A = \{a, b, c, d, e, f\}$:

1. $\#(\mathcal{P}(A)) =$
2. $\#\{s \in \mathcal{P}(A) : \{a, e\} \subseteq s\}) =$
3. $\#\{s \in \mathcal{P}(A) : \{a, e\} \subset s\}) =$

8. (*)

In axiomatic set theory, all entities are sets, and that includes numbers, which are represented by sets that are constructed in some special way. A common technique for building up natural "numbers" is called "von Neumann construction", and it works as follows. First, the number 0 (zero) is represented by the empty set $\emptyset$.

Starting from that, given any "number" n, the set representing the next number (that is, the number one greater than the previous one), let us call it n^+, is defined as:

$$n^+ = n \cup \{n\}$$

1. Construct the sets representing first few natural numbers:

   $0 = \emptyset$

   $1 = 0^+ =$ ____________________

   $2 = 1^+ =$ ____________________

   $3 = 2^+ =$ ____________________

2. How would one compare numbers in this representation? Suppose m and n are von-Neumann-constructed numbers as above:

   $m \leq n$ iff ____________________

3. Compute the maximum and the minimum of two von Neumann numbers:

   (a) $\max(m, n) =$ ____________________

   (b) $\min(m, n) =$ ____________________

4. Let's add two von Neumann numbers (we use the fact that every von Neumann number is either zero, or the "next number" to another number just one smaller):

$$\text{plus}(m, n) = \begin{cases} & \text{if} \quad n = \emptyset \\ & \text{if} \quad n = k^+ \end{cases}$$

Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e. including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $\mathbb{P}$ the prime numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k (k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- $[a, b],]a, b[,]a, b], [a, b[$ closed, open, and half-open intervals from a to b
- $\sum S$ sum of all elements of S
- $\prod S$ product of all elements of S
- $\bigcup S$ union of all elements in S
- $\bigcap S$ intersection of all elements in S
- $\bigcup_{a \in S} E(a)$, $\bigcap_{a \in S} E(a)$ generalized union / intersection of the sets computed for every a in S