Exercises 1 — sets

1.

Using set builder notation, define the following sets (you can use the usual sets of numbers as starting points, see the last page):

1. The set E of all even natural numbers (starting at 0).

   $E = \{2n : n \in \mathbb{N}\}$

2. Given the set $D = \{d \in \mathbb{N} : d \leq 9\}$, define the family of sets $\{L_d : d \in D\}$ such that L_d is the set of all natural numbers whose decimal representation ends with the digit d.

   $L_d = \{n \in \mathbb{N} : n \mod 10 = d\}$

3. The set T of natural numbers that are the product of exactly three different primes.

   $T = \{pqr : p, q, r \in \mathbb{P}, p \neq q, q \neq r, r \neq p\}$

4. The set X of pairs of natural numbers such that in each pair, the left-hand number is a square number, and the right-hand number is the cube of the same value that the left-hand number is the square of.

   $X = \{(k^2, k^3) : k \in \mathbb{N}\}$

2.

Given $A = \{a, b, c, d, e, f\}$, what is

1. $\#(\{s \in \mathcal{P}(A) : \#(s) = 0\}) = 1$
2. $\#(\{s \in \mathcal{P}(A) : \#(s) = 1\}) = 6$
3. $\#(\{s \in \mathcal{P}(A) : \#(s) = 2\}) = 15$
4. $\#(\{s \in \mathcal{P}(A) : \#(s) = 3\}) = 20$
5. $\#(\{s \in \mathcal{P}(A) : \#(s) = 4\}) = 15$
6. $\#(\{s \in \mathcal{P}(A) : \#(s) = 5\}) = 6$

7. $\#(\{s \in \mathcal{P}(A) : \#(s) = 6\}) = 1$

3.

Given $A = \{3, 4, 5, 6, 7, 8\}$, suppose we define the following sets

$$B = \left\{ \frac{a-b}{a+b} : a, b \in A \right\}$$

$$C = \left\{ \frac{a}{b} : a, b \in A \wedge a \geq b \right\}$$

Give the number of elements in these sets as follows:

1. $\#(B) =$ __27__
2. $\#(C) =$ __14__

4.

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$. For any non-empty set S of numbers, $\max S$ and $\min S$ are the largest and smallest numbers S, respectively.

1. $\#\{(a, b) \in A \times A : a \leq b\} = 55$
2. $\max \left\{ \frac{a}{b} : a, b \in A \right\} - \min \left\{ \frac{a}{b} : a, b \in A \right\} = 99/10$
3. $\min \left\{ \frac{a}{b} : a, b \in A, a > b \right\} - \max \left\{ \frac{a}{b} : a, b \in A, a \leq b \right\} = 1/9$

5.

Given $A = \{a \in \mathbb{N}^+ : a \leq 6\}$, let us define the following sets:

$$B = \left\{\frac{a}{b} : a, b \in A\right\}$$
$$C = \left\{\frac{a}{b} : a, b \in A, b|a\right\}$$
$$D = \left\{\frac{a}{b} : a, b \in A, a|b\right\}$$

Give the number of elements in these sets as follows:

1. $\#(B) =$ 23
2. $\#(C) =$ 6
3. $\#(D) =$ 6

6.

Given $A = \{1, 2, 3, 4, 5, 6, 7\}, B = \{3, 4, 5, 6, 7, 8\},\ C = \{ab : a \in A, b \in B\}$
$D = \{ab : a \in A, b \in B, a > b\}$, and $E = \{ab : a \in A, b \in A \cap B\}$:

1. $\#(C) =$ 27
2. $\#(D) =$ 10
3. $\#(E) =$ 23

7.

Given $A = \{a, b, c, d, e, f\}$:

1. $\#(\mathcal{P}(A)) =$ 64
2. $\#\{s \in \mathcal{P}(A) : \{a, e\} \subseteq s\}) =$ 16
3. $\#\{s \in \mathcal{P}(A) : \{a, e\} \subset s\}) =$ 15

8.

In axiomatic set theory, all entities are sets, and that includes numbers, which are represented by sets that are constructed in some special way. A common technique for building up natural "numbers" is called "von Neumann construction", and it works as follows. First, the number 0 (zero) is represented by the empty set $\emptyset$.

Starting from that, given any "number" n, the set representing the next number (that is, the number one greater than the previous one), let us call it n^+, is defined as:

$$n^+ = n \cup \{n\}$$

1. Construct the sets representing first few natural numbers:

   0 = $\emptyset$

   1 = $0^+ = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}$

   2 = $1^+ = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = \{0, 1\}$

   3 = $2^+ = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{0, 1, 2\}$

2. How would one compare numbers in this representation? Suppose m and n are von-Neumann-constructed numbers as above:

   $m \leq n$ iff $m \subseteq n$

3. Compute the maximum and the minimum of two von Neumann numbers:

   (a) $\max(m, n) = m \cup n$

   (b) $\min(m, n) = m \cap n$

4. Let's add two von Neumann numbers (we use the fact that every von Neumann number is either zero, or the "next number" to another number just one smaller):

$$\text{plus}(m, n) = \begin{cases} m & \text{if} \quad n = \emptyset \\ \text{plus}(m^+, k) & \text{if} \quad n = k^+ \end{cases}$$