Exercises 2 — relations

1.

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ and a family of relations
$R_i = \{(a, b) \in A \times A : \mod (b, a) = i\}$ for any $i \in \mathbb{N}$, with $\mod (b, a)$ the remainder when dividing positive integer b by positive integer a. So, for example,
$R_3 = \{(a, b) \in A \times A : \mod (b, a) = 3\}$.

1. $\#R_4 =$
2. $\#R_5 =$
3. $\#R_0 =$
4. $\#R_{10} =$
5. $R_3(7) =$
6. $R_1(2) =$
7. $R_1(A) =$
8. $R_3(A) =$

2.

With $A = \{n \in \mathbb{N}^+ : n \leq 20\}$ and $R = \{(a, b) \in A^2 : a|b\}$ compute the following images of R:

1. $R(6) =$

2. $R(7) =$

3. $R(2) =$

4. $R(\{2, 5\}) =$

3.

With $A = \{n \in \mathbb{N}^+ : n \leq 10\}$ and $R = \{(a, (a + b)) : a \in A, b \in A, a \perp b\}$ compute the following images of R:

1. $R(2) =$

2. $R(3) =$

3. $R(4) =$

4. $R(\{5, 7\}) =$

4.

With $A = \{1, 2, 3, 4, 5, 6, 7\}$, < the usual arithmetic order on A, and the relations $B = \{(a, b) \in A^2 : a < b\}$, and $C = \{(a, b) \in A^2 : a \perp b\}$ (the complement is supposed to be built with respect to $A \times A$):

1. $\#(B) =$
2. $\#(C) =$
3. $\#(\overline{B}) =$
4. $\#(B^{-1}) =$

5.

With $A = \{2, 3, 4, 5, 6, 7\}$, $R = \{(a, b) \in A^2 : a \perp b\}$, and $S = \{(a, b) \in A^2 : a | b\}$. We are looking at the composition $S \circ R$ in this task.

1. $\#(S \circ R) =$______
2. $S \circ R(2) =$____
3. $S \circ R(3) =$____
4. $S \circ R(6) =$____
5. $S \circ R(7) =$____
6. $S \circ R$ is ... (circle those that apply)

… reflexive on A	TRUE	FALSE
… symmetric	TRUE	FALSE
… transitive	TRUE	FALSE
… antisymmetric	TRUE	FALSE

7. R is ... (circle those that apply)

… reflexive on A	TRUE	FALSE
… symmetric	TRUE	FALSE
… transitive	TRUE	FALSE
… antisymmetric	TRUE	FALSE

6.

With $A = \{1, 2, 3, 4, 5, 6, 7\}$, < the usual arithmetic order on A, and $< \circ <$ its composition with itself:

1. $<(1) =$
2. $< \circ < (1) =$
3. $\{(a, b) \in A^2 : a(< \circ <)b\} =$
4. Is $< \circ <$ transitive? YES NO
5. Is $< \circ <$ reflexive on $A \times A$? YES NO

7.

With $A = [1, 7]$ in the **real numbers** $\mathbb{R}$, < the usual arithmetic order on A, and $< \circ <$ its composition with itself:

1. $<(1) =$
2. $< \circ < (1) =$

8.

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ and a relation $R_3 = \{(a, b) \in A \times A : \mod(b, a) = 3\}$.

Let $T = R_3 \circ R_3^{-1}$.

1. $\#T =$
2. $T(1) =$
3. $T(7) =$
4. $T(A) =$
5. T is ... (circle those that apply)

… reflexive	TRUE	FALSE
… symmetric	TRUE	FALSE
… transitive	TRUE	FALSE
… antisymmetric	TRUE	FALSE

Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e. including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $\mathbb{P}$ the prime numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k (k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- $\overline{R}$ of a relation R: its *complement*
- R^{-1} of a relation R: its *inverse*
- $R \circ S, f \circ g$ of relations and functions: their *composition*
- $R[X], f[X]$ *closure* of a set X under a relation R, a set of relations R, or a function f
- $[a, b],]a, b[,]a, b], [a, b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are *equinumerous*
- A^* for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, ε
- $\sum S$ sum of all elements of S
- $\prod S$ product of all elements of S
- $\bigcup S$ union of all elements in S
- $\bigcap S$ intersection of all elements in S
- $\bigcup_{a \in S} E(a)$, $\bigcap_{a \in S} E(a)$ generalized union / intersection of the sets computed for every a in S