Exercises 2 — relations

1.

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ and a family of relations
$R_i = \{(a, b) \in A \times A : \mod (b, a) = i\}$ for any $i \in \mathbb{N}$, with $\mod (b, a)$ the remainder when dividing positive integer b by positive integer a. So, for example,
$R_3 = \{(a, b) \in A \times A : \mod (b, a) = 3\}$.

1. $\#R_4 = 8$
2. $\#R_5 = 5$
3. $\#R_0 = 27$
4. $\#R_{10} = 0$
5. $R_3(7) = \{3, 10\}$
6. $R_1(2) = \{1, 3, 5, 7, 9\}$
7. $R_1(A) = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$
8. $R_3(A) = \{3, 7, 8, 9, 10\}$

The most frequent question I get about these kinds of tasks is whether there is a “shortcut”. In this case, one way to go about answering this question is to construct a little table representing the modulo operation for numbers between 1 and 10. Note that the table contains a number of regularities, and it can be constructed very easily. Just follow the first few rows to see how this is built.

		b									
	b mod a	1	2	3	4	5	6	7	8	9	10
a	1	0	0	0	0	0	0	0	0	0	0
	2	1	0	1	0	1	0	1	0	1	0
	3	1	2	0	1	2	0	1	2	0	1
	4	1	2	3	0	1	2	3	0	1	2
	5	1	2	3	4	0	1	2	3	4	0
	6	1	2	3	4	5	0	1	2	3	4
	7	1	2	3	4	5	6	0	1	2	3
	8	1	2	3	4	5	6	7	0	1	2
	9	1	2	3	4	5	6	7	8	0	1
	10	1	2	3	4	5	6	7	8	9	0

Once you have this table, answering 2.1 – 2.8 becomes pretty straightforward.

2.

With $A = \{n \in \mathbb{N}^+ : n \leq 20\}$ and $R = \{(a, b) \in A^2 : a|b\}$ compute the following images of R:

1. $R(6) = \{6, 12, 18\}$
2. $R(7) = \{7, 14\}$
3. $R(2) = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$
4. $R(\{2, 5\}) = \{2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18, 20\}$

3.

With $A = \{n \in \mathbb{N}^+ : n \leq 10\}$ and $R = \{(a, (a + b)) : a \in A, b \in A, a \perp b\}$ compute the following images of R:

1. $R(2) =$ {3, 5, 7, 9, 11}
2. $R(3) =$ {4, 5, 7, 8, 10 11, 13}
3. $R(4) =$ {5, 7, 9, 11, 13}
4. $R(\{5, 7\}) =$ {6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17}

4.

With $A = \{1, 2, 3, 4, 5, 6, 7\}$, $<$ the usual arithmetic order on A, and the relations $B = \{(a, b) \in A^2 : a < b\}$, and $C = \{(a, b) \in A^2 : a \perp b\}$ (the complement is supposed to be built with respect to $A \times A$):

1. $\#(B) =$ 21
2. $\#(C) =$ 35
3. $\#(\overline{B}) =$ 28
4. $\#(B^{-1}) =$ 21

5.

With $A = \{2, 3, 4, 5, 6, 7\}$, $R = \{(a, b) \in A^2 : a \perp b\}$, and $S = \{(a, b) \in A^2 : a | b\}$. We are looking at the composition $S \circ R$ in this task.

1. $\#(S \circ R) =$ 25
2. $S \circ R(2) =$ {3, 5, 6, 7}
3. $S \circ R(3) =$ {2, 4, 5, 6, 7}
4. $S \circ R(6) =$ {5, 7}
5. $S \circ R(7) =$ {2, 3, 4, 5, 6}
6. $S \circ R$ is ... (circle those that apply)
   … reflexive on A TRUE [✓ FALSE]
   … symmetric TRUE [✓ FALSE]
   … transitive TRUE [✓ FALSE]
   … antisymmetric TRUE [✓ FALSE]

7. R is ... (circle those that apply)
   … reflexive on A TRUE [✓ FALSE]
   … symmetric [✓ TRUE] FALSE
   … transitive TRUE [✓ FALSE]
   … antisymmetric TRUE [✓ FALSE]

6.

With $A = \{1, 2, 3, 4, 5, 6, 7\}$, $<$ the usual arithmetic order on A, and $< \circ <$ its composition with itself:

1. $<(1) =$ {2, 3, 4, 5, 6, 7}
2. $< \circ < (1) =$ {3, 4, 5, 6, 7}
3. $\{(a, b) \in A^2 : a(< \circ <)b\} =$ {(1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 4), (2, 5), (2, 6), (2, 7), (3, 5), (3, 6), (3, 7), (4, 6), (4, 7), (5, 7)}
4. Is $< \circ <$ transitive? [✓ YES] NO
5. Is $< \circ <$ reflexive on $A \times A$? YES [✓ NO]

7.

With $A = [1, 7]$ in the **real numbers** $\mathbb{R}$, $<$ the usual arithmetic order on A, and $< \circ <$ its composition with itself:

1. $<(1) =$]1, 7]
2. $< \circ < (1) =$]1, 7]

8.

Suppose $A = \{n \in \mathbb{N} : 1 \leq n \leq 10\}$ and a relation $R_3 = \{(a,b) \in A \times A : \quad \operatorname{mod}\,(b,a) = 3\}$.

Let $T = R_3 \circ R_3^{-1}$.

Here, too, things become a lot simpler once you have explicitly constructed the extensions of the relations involved. So for reference:

$R_3 = \{(4,3),(4,7),(5,3),(5,8),(6,3),(6,9),(7,3),(7,10),(8,3),(9,3),(10,3)\}$

$R_3^{-1} = \{(3,4),(7,4),(3,5),(8,5),(3,6),(9,6),(3,7),(10,7),(3,8),(3,9),(3,10)\}$

$T = \{(3,3),(7,7),(8,8),(9,9),(10,10),(3,7),(3,8),(3,9),(3,10),(10,3),(9,3),(8,3),(7,3)\}$

1. $\#T =$ 13
2. $T(1) =$ {}
3. $T(7) =$ {3, 7}
4. $T(A) =$ {3, 7, 8, 9, 10}
5. T is ... (circle those that apply)

… reflexive	TRUE	[✓ FALSE]
… symmetric	[✓ TRUE]	FALSE
… transitive	TRUE	[✓ FALSE]
… antisymmetric	TRUE	[✓ FALSE]