Exercises 3 — functions

1.

Define two sets A and B, as well as a function $f : A \longrightarrow B$, such that f is **surjective** and **not injective**.

$A = \{a, b\}$

$B = \{x\}$

$f = \{(a, x), (b, x)\}$

2.

Suppose $f : x \mapsto (x + 1)^2$ with domain A and codomain B.

1. If $A = [-2, 2]$, what is B so that f is surjective? $B =$ [0, 9]
2. If $B = [0, 4]$, what is a set A such that f is bijective? $A =$ [-1, 1]
3. If $B = [0, 4]$, what is the largest possible set A? $A =$ [-3, 1]

3.

Assume you have a surjection $s : A \twoheadrightarrow B$ and an injection $j : B \hookrightarrow C$.

1. Their composition $j \circ s$ is not always injective. Show this using a counterexample, by giving definitions for A, B, C, s, and j, and demonstrating that their composition $j \circ s$ is not injective.

$A = \{a_1, a_2\}$

$B = \{b\}$

$C = \{c_1, c_2\}$

$s = \{(a_1, b), (a_2, b)\}$

$j = \{(b, c_1)\}$

Now demonstrate that the composition $j \circ s$ is not injective:

$$j \circ s(a_1) = j(s(a_1)) = j(b) = c_1$$
$$j \circ s(a_2) = j(s(a_2)) = j(b) = c_1$$

The composition is not injective because $a_1 \neq a_2$, yet $j \circ s(a_1) = j \circ s(a_2)$.

2. Their composition $j \circ s$ is also not always surjective. Show this using a counterexample, by giving definitions for A, B, C, s, and j, and demonstrating that their composition $j \circ s$ is not surjective.

$A = \{a_1, a_2\}$

$B = \{b\}$

$C = \{c_1, c_2\}$

$s = \{(a_1, b), (a_2, b)\}$

$j = \{(b, c_1)\}$

Now demonstrate that the composition $j \circ s$ is not surjective:

$j \circ s(A) = j \circ s(\{a_1, a_2\}) = \{j \circ s(a_1), j \circ s(a_2)\} = \{c_1, c_1\} = \{c_1\} \neq C$

4.

Assume you have an injection $j : A \hookrightarrow B$ and a surjection $s : B \twoheadrightarrow C$.

1. Their composition is not always injective. Show this using a counterexample, by giving definitions for A, B, C, j, and s, and demonstrating that their composition $s \circ j$ is not injective.

$A = \{a_1, a_2\}$

$B = \{b_1, b_2\}$

$C = \{c\}$

$j = \{(a_1, b_1), (a_2, b_2)\}$

$s = \{(b_1, c), (b_2, c)\}$

Now demonstrate that the composition $s \circ j$ is not injective:

$s \circ j(a_1) = s(j(a_1)) = s(b_1) = c$
$s \circ j(a_2) = s(j(a_2)) = s(b_2) = c$

The composition is not injective because $a_1 \neq a_2$, yet $s \circ j(a_1) = s \circ j(a_2)$.

2. Their composition is not always surjective. Show this using a counterexample, by giving definitions for A, B, C, j, and s, and demonstrating that their composition $s \circ j$ is not surjective.

$A = \{a\}$

$B = \{b_1, b_2\}$

$C = \{c_1, c_2\}$

$j = \{(a, b_1)\}$

$s = \{(b_1, c_1), (b_2, c_2)\}$

Now demonstrate that the composition $s \circ j$ is not surjective:

$s \circ j(A) = s \circ j(\{a\}) = \{s \circ j(a)\} = \{c_1\} \neq C$

5.

Suppose you have **injections** $f : A \hookrightarrow B$ and $g : A \hookrightarrow B$, as well as a **non-empty** set $S \subset A$ (note that S is a **proper** subset of A). Now let's define a function $h : A \longrightarrow B$ as follows:

$$h : x \mapsto \begin{cases} f(x) & \text{for } x \in S \\ g(x) & \text{for } x \notin S \end{cases}$$

This function is not, in general, injective.
Whether it is injective depends on the definitions of A, B, f, g, and S.

1. Give definitions for A, B, f, g, and S such that the h above is injective.

   $A = \{a, b\}$

   $B = \{x, y\}$

   $f = \{(a, x), (b, y)\}$

   $g = \{(a, x), (b, y)\}$

   $S = \{a\}$

2. Give definitions for A, B, f, g, and S such that the h above is **not** injective.

   $A = \{a, b\}$

   $B = \{x, y\}$

   $f = \{(a, x), (b, y)\}$

   $g = \{(a, y), (b, x)\}$

   $S = \{a\}$

3. Give a general formal criterion, depending only on A, B, f, g, and S (not necessarily all of them), that defines the condition under which h is injective. (Hint: Remember, f and g are already injective.)

   h is injective if and only if $f(S) \cap g(A \setminus S) = \emptyset$

Note: You are **not** supposed to reiterate the definition of injectivity for h, but rather give an expression involving at most A, B, f, g, and S (but **not** h) that is true if an only if they lead to an injective h.

6.

Suppose we have a set A that is **totally ordered** by a strict total order relation $<$ on A. A function $f : A \longrightarrow A$ is called *strictly monotonic* iff for any $a, b \in A$ it is the case that $a < b$ implies that $f(a) < f(b)$.

1. Show that such a strictly monotonic function f is always injective.

   It is to show that $x \neq y$ implies $f(x) \neq f(y)$
   Since $x \neq y$, and A is totally ordered, then either $x < y$ or $y < x$.
   If the former, then $f(x) < f(y)$ and therefore $f(x) \neq f(y)$.
   If the latter, correspondingly.

   You need to show that $x \neq y$ implies $f(x) \neq f(y)$, so $x \neq y$ is the starting point of the proof. To get from there to $x < y$ (or equivalently $y < x$), you need to invoke the *total* order property of $<$.

2. A strictly monotonic function is not, however, necessarily surjective. Demonstrate this by giving a counterexample.

   $$\begin{aligned} f: \ & \mathbb{R} \longrightarrow \mathbb{R} \\ & x \mapsto 2^x \end{aligned}$$

   f is strictly monotonic, but not surjective.

7.

Define **one** set A, as well as a function $f : A \longrightarrow A$, such that f is **surjective** and **not injective**.

$A = \mathbb{N}$

$$f : a \mapsto \begin{cases} 0 & \text{if } a = 0 \\ a - 1 & \text{if } a > 0 \end{cases}$$

The crux here is that A has to be infinite. This is a lead-in to the next lecture.