Exercises 4 — infinity

1.

Recall that a set is infinite iff it is equinumerous to a proper subset of itself. Show that the set of all non-negative rational numbers (i.e. including 0) $\mathbb{Q}_0^+$ is infinite in two steps:

1. Define a **proper** subset of $A \subset \mathbb{Q}_0^+$ – to make this a little more interesting, this subset should be wholly contained between two rational numbers $a, b \in \mathbb{Q}$, i.e. for all $r \in A$, it should be true that $a \leq r \leq b$.

$$A =$$

2. Now construct a **bijection** $f : \mathbb{Q}_0^+ \longleftrightarrow A$

$$f : q \mapsto$$

Note: You do **not** need to *prove* that your f is bijective. For this task it is sufficient if the function you specify has that property.

2.

In lecture 4, slide 18, it is said that $\#(\mathbb{N}) \leq \#(2^{\mathbb{N}})$. Show this.

Hint: You need to define something in order to prove this, and then show that the thing you defined has a certain property. Go step by step:

1. Start by writing down what you need to define and what property it must have.
2. Then define it.
3. Then show that it has the property.

3.

Show that $[0, 1] \ \sim \ ]0, 1[$ in the real numbers, i.e. that the open interval from 0 to 1 is equinumerous to the closed one from 0 to 1.

This becomes fairly straightforward if you use CSB. If that is not challenging enough, try proving this *without* using CSB, i.e. by constructing an actual bijection between the two intervals.

4.

On slide 21 of lecture 4, it is claimed that for any set A, we have $\#A < \#\mathcal{P}(A)$, i.e. the power set is always strictly larger than the set. (This then implies the existence of infinitely many transfinite cardinal numbers.)

Show that $\#A < \#\mathcal{P}(A)$ as follows. (This is slightly challenging, so do not feel bad if you get stuck. Instead, post a question in Piazza.)

The path here is via a slight generalization of Cantor's diagonal proof. Basically, you need to show that for any set A, there is no surjective function $f : A \twoheadrightarrow \mathcal{P}(A)$.

In the diagonal proof in the lecture, Cantor assumed that we have such a surjective function from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$, and then constructs an element of the codomain of that function that it cannot possibly map to – which contradicts the assumption that it was surjective and thus implies that there cannot be a surjective function from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$.

In the diagonal proof as shown in the lecture, the domain of the function was $\mathbb{N}$, and the codomain infinite sequences of 0s and 1s, each of which is really a function $s : \mathbb{N} \longrightarrow \{0,1\}$, which is why the codomain of f is $\{0,1\}^{\mathbb{N}}$, or equivalently $2^{\mathbb{N}}$.

At the bottom of slide 15, there is a comment that the set $\{0,1\}^{\mathbb{N}}$ of infinite sequences of 0s and 1s can also be thought of as the powerset $\mathcal{P}(\mathbb{N})$ of the natural numbers. This is because you could represent an infinite sequence of 0s and 1s $s : \mathbb{N} \longrightarrow \{0,1\}$ simply as the set of all natural numbers for which s is 1, i.e. $\{k \in \mathbb{N} : s(k) = 1\}$. Conversely, for any set $A \subseteq \mathbb{N}$ you could construct a sequence function $s : \mathbb{N} \longrightarrow \{0,1\}$ such that

$$s(k) = \begin{cases} 1 & \text{for } s \\ 0 & \text{for } s \notin A \end{cases}.$$

In other words, there is a 1:1 correspondence between the two. But this means that you could think of Cantor's diagonal proof already as a special case of what we want to show above, namely $\#\mathbb{N} < \#\mathcal{P}(\mathbb{N})$.

Armed with this insight, you now need to generalize this just a tiny bit. Suppose there is a function

$$f : A \longrightarrow \mathcal{P}(A)$$

You need to use it to build a set $\overline{D} \in \mathcal{P}(A)$ (or simply $\overline{D} \subseteq A$) such that $\overline{D} \notin \mathrm{range}\ f$. If such a $\overline{D}$ exists for any f, it means there cannot be a surjective f and thus there cannot be a bijection. In the proof in the lecture, we arrived at this $\overline{D}$ by first constructing the diagonal sequence D from f, and then inverting it. We shall do the same now here for any set A.

1. First, define a set $D \in \mathcal{P}(A)$ (or $D \subseteq A$) that corresponds to the interpretation of the diagonal sequence D from the lecture as a set.

   $D = \{a \in A : \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$

2. Now define $\overline{D}$, corresponding to the inverse of the diagonal sequence from the proof as shown in the lecture:

   $\overline{D} = \{a \in A : \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$

3. All that is left is to show is that given any function f, the corresponding $\overline{D} \notin \mathrm{range}\ f$.

   You can show this by demonstrating that for any $a \in A$, $f(a) \neq \overline{D}$. Remember that two sets are different iff there is at least one element that is in one of them, but not the other:

   $f(a) \neq \overline{D}$ because…