Exercises 4 — infinity

1.

Recall that a set is infinite iff it is equinumerous to a proper subset of itself. Show that the set of all non-negative rational numbers (i.e. including 0) $\mathbb{Q}_0^+$ is infinite in two steps:

1. Define a **proper** subset of $A \subset \mathbb{Q}_0^+$ – to make this a little more interesting, this subset should be wholly contained between two rational numbers $a, b \in \mathbb{Q}$, i.e. for all $r \in A$, it should be true that $a \leq r \leq b$.

   $A =]0, 1]_{\mathbb{Q}_0^+}$

2. Now construct a **bijection** $f : \mathbb{Q}_0^+ \longleftrightarrow A$

   $f : q \mapsto \frac{1}{q+1}$

Note: You do **not** need to *prove* that your f is bijective. For this task it is sufficient if the function you specify has that property.

2.

In lecture 4, slide 18, it is said that $\#(\mathbb{N}) \leq \#(2^{\mathbb{N}})$. Show this.

Hint: You need to define something in order to prove this, and then show that the thing you defined has a certain property. Go step by step:

1. Start by writing down what you need to define and what property it must have.
2. Then define it.
3. Then show that it has the property.

$\#(\mathbb{N}) \leq \#(2^{\mathbb{N}})$ is true if there is an injection $j : \mathbb{N} \hookrightarrow 2^{\mathbb{N}}$. Let $j : n \mapsto s_n$, where s_n is the sequence where the n-th entry is 1, all others are 0, more formally: $s_n : i \mapsto \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$

Since $s_n \neq s_m$ if $n \neq m$, the function is injective.

3.

Show that $[0,1] \sim\ ]0,1[$ in the real numbers, i.e. that the open interval from 0 to 1 is equinumerous to the closed one from 0 to 1.

This becomes fairly straightforward if you use CSB. If that is not challenging enough, try proving this *without* using CSB, i.e. by constructing an actual bijection between the two intervals.

Using CSB, we need to construct two injections, $f : [0,1] \hookrightarrow]0,1[$ and $g :]0,1[\hookrightarrow [0,1]$.

The injection g simply the identity, $g : x \mapsto x$.

The injection f maps its domain to an interval somewhere inside $]0,1[$ after suitably scaling it. For example, $f : x \mapsto 0.1 + \frac{x}{2}$.

Without CSB, we need to do something "clever" about the endpoints 0 and 1 of the closed interval when constructing the bijection $b : [0,1] \longleftrightarrow]0,1[$.

The trick is to pick an injection $x : \mathbb{N} \hookrightarrow]0,1[$, so we get a sequence of numbers $x(0), x(1), x(2), \ldots$ without repetition.

Now we can define $b : [0,1] \longleftrightarrow]0,1[$ as follows:

$$b : r \mapsto \begin{cases} x(0) & \text{if } r = 0 \\ x(1) & \text{if } r = 1 \\ x(k+2) & \text{if } r = x(k) \\ r & \text{if } r \in]0,1[\setminus \operatorname{range} x \end{cases}$$

Convince yourself that this is, indeed, a bijection.

This is a version of a simple case of Hilbert's hotel, a common metaphor to explain some of the counterintuitive stuff about infinity. Look it up and see if you can make the connection.

Does this technique only work for the interval of real numbers, or could we also use it, for example, for intervals of rational numbers, and why or why not?

4.

On slide 21 of lecture 4, it is claimed that for any set A, we have $\#A < \#\mathcal{P}(A)$, i.e. the power set is always strictly larger than the set. (This then implies the existence of infinitely many transfinite cardinal numbers.)

Show that $\#A < \#\mathcal{P}(A)$ as follows. (This is slightly challenging, so do not feel bad if you get stuck. Instead, post a question in Piazza.)

The path here is via a slight generalization of Cantor's diagonal proof. Basically, you need to show that for any set A, there is no surjective function $f : A \twoheadrightarrow \mathcal{P}(A)$.

In the diagonal proof in the lecture, Cantor assumed that we have such a surjective function from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$, and then constructs an element of the codomain of that function that it cannot possibly map to – which contradicts the assumption that it was surjective and thus implies that there cannot be a surjective function from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$.

In the diagonal proof as shown in the lecture, the domain of the function was $\mathbb{N}$, and the codomain infinite sequences of 0s and 1s, each of which is really a function $s : \mathbb{N} \longrightarrow \{0,1\}$, which is why the codomain of f is $\{0,1\}^{\mathbb{N}}$, or equivalently $2^{\mathbb{N}}$.

At the bottom of slide 15, there is a comment that the set $\{0,1\}^{\mathbb{N}}$ of infinite sequences of 0s and 1s can also be thought of as the powerset $\mathcal{P}(\mathbb{N})$ of the natural numbers. This is because you could represent an infinite sequence of 0s and 1s $s : \mathbb{N} \longrightarrow \{0,1\}$ simply as the set of all natural numbers for which s is 1, i.e. $\{k \in \mathbb{N} : s(k) = 1\}$. Conversely, for any set $A \subseteq \mathbb{N}$ you could construct a sequence function $s : \mathbb{N} \longrightarrow \{0,1\}$ such that

$$s(k) = \begin{cases} 1 & \text{for } s \\ 0 & \text{for } s \notin A \end{cases}.$$

In other words, there is a 1:1 correspondence between the two. But this means that you could think of Cantor's diagonal proof already as a special case of what we want to show above, namely $\#\mathbb{N} < \#\mathcal{P}(\mathbb{N})$.

Armed with this insight, you now need to generalize this just a tiny bit. Suppose there is a function

$$f : A \longrightarrow \mathcal{P}(A)$$

You need to use it to build a set $\overline{D} \in \mathcal{P}(A)$ (or simply $\overline{D} \subseteq A$) such that $\overline{D} \notin \text{range } f$. If such a $\overline{D}$ exists for any f, it means there cannot be a surjective f and thus there cannot be a bijection. In the proof in the lecture, we arrived at this $\overline{D}$ by first constructing the diagonal sequence D from f, and then inverting it. We shall do the same now here for any set A.

1. First, define a set $D \in \mathcal{P}(A)$ (or $D \subseteq A$) that corresponds to the interpretation of the diagonal sequence D from the lecture as a set.

   $D = \{a \in A :$ ____________________$\}$

   $$D = \{a \in A : a \in f(a)\}$$

2. Now define $\overline{D}$, corresponding to the inverse of the diagonal sequence from the proof as shown in the lecture:

   $\overline{D} = \{a \in A :$ ____________________$\}$

   $$\overline{D} = \{a \in A : a \notin f(a)\}$$

3. All that is left is to show is that given any function f, the corresponding $\overline{D} \notin \text{range } f$.

   You can show this by demonstrating that for any $a \in A$, $f(a) \neq \overline{D}$. Remember that two sets are different iff there is at least one element that is in one of them, but not the other:

   $f(a) \neq \overline{D}$ because… if $a \in f(a)$ then $a \notin \overline{D}$ and if $a \notin f(a)$ then $a \in \overline{D}$.