# EDAA40 – Seminar 6

10 May 2019

## 1

Suppose we have a graph $(V, E)$ with vertices $V$ and edges $E \subseteq V \times V$, where the $v \in V$ represent, for example, cities and each $(v_1, v_2) \in E$ is a road from $v_1$ to $v_2$. (This graph is directed, i.e. if $v_1 \neq v_2$, then $(v_1, v_2)$ and $(v_2, v_1)$, if present, are distinct roads, each of which can only be traveled in one direction.)

We also have two labeling functions:

- $F : E \longrightarrow \mathbb{Q}^+$, labeling each edge $(v_1, v_2) \in E$ with the number of fuel units required to take the road represented by that edge from its start $v_1$ to its end $v_2$, as well as
- $T : E \longrightarrow \mathbb{Q}^+$, labeling each edge $(v_1, v_2) \in E$ with the number of time units it takes to go from $v_1$ to $v_2$ using the direct road represented by that edge between them.

In the following, you may use functions given or defined (by you) in previous subtasks, irrespective of whether your definition is correct.

1. Define a function $r : V \times \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \longrightarrow \mathcal{P}(V)$ such that for any vertex $v \in V$ and any non-negative rational numbers $f, t \in \mathbb{Q}_0^+,\ r(v, f, t)$ is the set of all vertices that can be reached from $v$ using **at most** $f$ units of fuel **and at most** $t$ units of time.

   Keep in mind: If going from one vertex to another involves using multiple edges, the fuel and time for the entire path is the sum of the fuel and time labels of the individual edges in the path. Also, since it does not take any fuel or time to go from a place to itself, for any $v \in V$ and any $f, t \in \mathbb{Q}_0^+$, it is always the case that $v \in r(v, f, t)$.

$$r : V \times \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \longrightarrow \mathcal{P}(V)$$

$$v, f, t \mapsto$$

2. Both fuel and time are costs of different kinds. Suppose you are given a cost function $C : \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \longrightarrow \mathbb{Q}_0^+$ that combines these two factors somehow into a single non-negative number, such that for any $f, t \in \mathbb{Q}_0^+$, with $f$ the fuel and $t$ the time as before, $C(f, t)$ is the combined *cost*. The details of $C$ do not matter here, but it has the property that

$$C(f_1 + f_2, t_1 + t_2) = C(f_1, t_1) + C(f_2, t_2).$$

This task is about defining a function $c : V \times V \longrightarrow \mathbb{Q}_0^+ \cup \{\infty\}$ such that for any two vertices $v_1, v_2 \in V$, the value $c(v_1, v_2)$ is the minimal combined cost for going from $v_1$ to $v_2$ following the edges in our graph, adding the costs of the individual edges along the way. The cost is $\infty$ iff we cannot get from $v_1$ to $v_2$ in this way, that is iff there is no path from $v_1$ to $v_2$ in the graph.

This is done using a helper function $c' : V \times \mathcal{P}(V) \times V \longrightarrow \mathbb{Q}_0^+ \cup \{\infty\}$, which keeps track of vertices that have not been visited yet. It is used as follows:

$$c(v, w) = c'(v, V \setminus \{v\}, w)$$

Now define $c'$ recursively:

$c' : V \times \mathcal{P}(V) \times V \longrightarrow \mathbb{Q}_0^+ \cup \{\infty\}$

$v, X, w \mapsto$

Note:
For this task, you may treat $\infty$ as a number greater than any rational number, and you can compare it, add to it, and so forth. You might also want to use a minimum function $\min$ that computes the minimum of a set of numbers, so e.g. $\min\{3, 2, 7\} = 2$, $\min\{5, \infty, 17\} = 5$, and also $\min \emptyset = \infty$.

3. In order to ensure that $c'$ terminates, we require a **well-founded strict order** $\prec$ of its arguments, such that for any $(v, X, w)$ that $c'$ is called on, it will only ever call itself on $(v', X', w') \prec (v, X, w)$. Define such an order:

$$(v', X', w') \prec (v, X, w) \iff$$

Hint: A correct answer to this question must have three properties.
1. It must be a strict order.
2. It must be well-founded, i.e. there cannot be an infinite descending chain in that order.
3. Your definition of $c'$ must conform to it, i.e. any recursive call in it must be called on a smaller (according to the order) triple of arguments.

4. Define the function $R : V \longrightarrow \mathcal{P}(V)$ that maps any vertex to the set of vertices that can be reached by following a path through the graph, including the empty path, so for any $v \in V$ it is always the case that $v \in R(v)$.

$$R : v \mapsto$$

5. Define the set $S \subseteq E$ of edges $(v, w)$ for which there exists an alternative path through the graph from $v$ to $w$ that has lower combined cost than the edge $(v, w)$ itself.

$$S =$$

6. Prove that the cost combination function $C$ above has the property that

$$C(0,0) = 0$$

7. We want to show that the cost combination function $C$ is (weakly) monotonic in both its arguments, that is, that it has the following two properties

   (a) $f \leq f' \Rightarrow C(f,t) \leq C(f',t)$

   (b) $t \leq t' \Rightarrow C(f,t) \leq C(f,t')$

   Prove (a). (The proof for (b) is analogous, so we can skip it.)