Hard and easy problems

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A very brief compendium of various computational problems whose complexity is considered well-known.

Some Problems in P

You can freely use that the following problems are in P, with the given running times.

Linear programming. Given an $m \times n$ matrix A, vectors $b \in R^m$ and $c \in R^n$, find vector $x \in R^n$ minimizing $c^t x \colon x \geq 0, Ax \geq b$.[1]

2-Sat. Given a boolean formula in conjunctive normal form with m clauses, with exactly two literals per clause, does the formula have a satisfying assignment? Time $O(m)$.[2]

Maximum Flow, Minimum Cut. Given directed graph with edge capacities, source and terminal vertex. What is the largest flow from the source to the terminal respecting the capacities? What is the smallest capacity of a cut separating the source from the terminal? Time $O(ms)$ where s is the size of the max flow [Ford–Fulkerson], or $O(nm^2)$ [Edmonds–Karp].[3]

Shortest Path. Given a directed, weighted graph with no negative cycles, find a shortest path between two given vertices. Time $O(nm)$ [Bellman–Ford] or $O(m + n \log n)$ [Dijkstra, assuming no negative weights]. [4]

Maximum Matching. Given a graph, find a matching (a subset of edges without common endvertices) of maximal size. Time $O(n^4)$ [Edmonds] or $O(nm)$ (for bipartite graphs).[5]

Determinant. Given $n \times n$ matrix A, compute $\det A$. Running time $O(n^3)$ [Gaussian elimination].

Matrix product. Given $n \times n$ matrices A and B, compute AB. Running time $O(n^3)$ or $O(n^{2.373})$ (Coppersmith-Winograd).

Prime. Decide if a given n-digit number is prime. Time $O(n^{12})$ [AKS].

[1] *Integer* Linear Programming is NP-hard.

[2] 3-Sat is NP-hard. Max 2-Sat is NP-hard.

[3] Maximum Cut is NP-hard.

[4] When negative cycles are allowed, problem is equivalent to Longest Path, which is NP-hard because it reduces from Hamiltionan Path.

[5] NP-hard in *hypergraphs*, even tripartite; reduces from 3D Matching.

Some NP-hard Problems

You can freely use that the following problems are NP-hard:

Circuit-Sat. Given Boolean circuit, are there input values that make the circuit output True?

3-Sat. Given Boolean formula in conjunctive normal form, with exactly three literals per clause, does the formula have a satisfying assignment?

Max 2-Sat. Given Boolean formula in conjunctive normal form, with exactly two literals per clause, what is the largest number of clauses that can be satisfied by an assignment?

Max Independent Set. Given undirected graph G, what is the size of the largest subset of vertices in G that have no edges among them?

Max Clique. Given undirected graph G, what is the size of the largest complete subgraph of G?

Min Vertex Cover. Given undirected graph G, what is the size of the smallest subset of vertices that touch every edge in G?

Min Set Cover. Given collection of subsets $S_1, \ldots, S_m$ of a set U, what is the size of the smalles subcollection whose union is U?

Min Hitting Set. Given collection of subsets $S_1, \ldots, S_m$ of a set U, what is the size of the smallest subset of U that intersects every S_i?

Vertex 3-Colouring. Given undirected graph G, can its vertices be colored with three colors, so that every edge touches vertices with two different colors?

Maxcut. Given a graph G, what is the size (number of edges) of the largest bipartite subgraph of G?

Hamiltonian Cycle/Path.. Given a graph G, is there a cycle/path in G that visits every vertex exactly once?

Traveling Salesman. Given a graph G with weighted edges, what is the minimum total weight of any Hamiltonian path/cycle in G?

Integer Programming. Given $m \times n$ matrix A, vectors $b \in R^m$ and $c \in R^n$, find integer vector $x \in Z^n$ minimizing $c^t x\colon x \geq 0, Ax \geq b$.[6]

[6] If instead x ranges over R^n, the problem is Linear Programming, in P.

Subset Sum. Given a set X of positive integers and integer k, does X have a subset whose elements sum to k?

Partition. Given a set X of positive integers, can X be partitioned into two subsets with the same sum?

3D Matching. Given n elements paritioned into three disjoint sets V_1, V_2, V_3 and family of triples (v_1, v_2, v_3) with $v_i \in V_i$, find subset of triples such that every element appears exactly once.[7]

[7] "2D Matching" is bipartite matching, in P.