EDAA40 Exam

26 August 2017

Model solutions are inserted in blue, additional comment is in red.

Instructions

Things you CAN use during the exam.
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam.
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favorable assumption about what you intended to write.

A sheet with common symbols and notations is attached at the end.

Good luck!

1	2	3	4	5	6	7	8	total
20	6	10	4	10	10	16	20	96

points required for 3: 50

points required for 4: 65

points required for 5: 84

1 [20 p]

For the following sets of numbers, specify the smallest and the largest numbers, write NONE if there is no smallest or largest number, or EMPTY (in one of the two columns) if the set is empty.

All intervals are supposed to be intervals in the real numbers, $\mathbb{R}$. Similarly, all relations and operators are on the real numbers, unless explicitly stated otherwise.

set	smallest element	largest element
$\{x \in \mathbb{Z} : x > 4 \land x < 2\}$	EMPTY	
$\{1, 2, 3, 4\}$	1	4
$]1, 4]$	NONE	4
$\bigcap_{i \in \mathbb{N}+} \left[0, \frac{1}{i}\right] = \{0\}$	0	0
$\bigcap_{i \in \mathbb{N}+} \left]0, \frac{1}{i}\right] = \emptyset$	EMPTY	
$\bigcap_{i \in \mathbb{N}+} \left[\frac{-1}{i}, \frac{1}{i}\right] = \{0\}$	0	0
$\bigcap_{i \in \mathbb{N}+} \left]\frac{-1}{i}, 1\right] = [0, 1]$	0	1
$\mathrm{sqrt}([0, 0.1]) = [0, \sqrt{0.1}]$	0	$\sqrt{0.1}$
$\mathrm{sqrt}[[0, 0.1]] = [0, 1[$	0	NONE
$\mathrm{sqrt}[]0, 0.1]] =]0, 1[$	NONE	NONE
$\bigcup_{i \in \mathbb{N}+} \left]\frac{1}{i+1}, \frac{1}{i}\right[=]0, 1[$	NONE	NONE
$\mathrm{inv}(]0, 0.1]) = [10, +\infty[$	10	NONE
$\mathrm{inv}[]0, 0.1]] =]0, 0.1] \cup [10, +\infty[$	NONE	NONE

$\mathrm{sqrt} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is the positive square root function, i.e. for every non-negative real number a, $\mathrm{sqrt}(a)$ is the non-negative real number such that $a = \mathrm{sqrt}(a) \cdot \mathrm{sqrt}(a)$.

$\mathrm{inv} : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R} \setminus \{0\}$ is the inversion function, defined by $\mathrm{inv} : r \mapsto \frac{1}{r}$.

When looking at intersections of an infinite number of sets, it is important to keep in mind that any value that is an element of the intersection must be an element of each and every one of those sets. Take for example $\bigcap_{i\in\mathbb{N}^+}\left[0,\frac{1}{i}\right]$. If that intersection contained any positive real number in addition to 0, it would mean that there is a real number $r > 0$, such that $r \in \left[0,\frac{1}{i}\right]$ for every $i \in \mathbb{N}^+$. It's easy to see that this cannot be the case: for any $r > 0$, there is some natural number $k > \frac{1}{r}$, which means that $\frac{1}{k} < r$, and so $r \notin \left[0,\frac{1}{k}\right]$, and so r cannot be in the intersection.

2 [6p]

Define two sets A and B, as well as a function $f : A \longrightarrow B$, such that f is **surjective** and **not injective**.

$A = \{a, b\}$

$B = \{x\}$

$f : a \mapsto x$

Of course, if you already found the answer to the next question, you could "reuse" it here by simply choosing to make A and B the same set.

The idea was to start with an easier question to get you to think about surjectivity and injectivity in a simpler setting first, before tackling the harder problem.

3 [10p]

Define **one** set A, as well as a function $f : A \longrightarrow A$, such that f is **surjective** and **not injective**.

$A = \mathbb{N}$

$$f : a \mapsto \begin{cases} 0 & \text{if } a = 0 \\ a - 1 & \text{otherwise} \end{cases}$$

The crux here is that A has to be infinite.

4 [4p]

Suppose there is a function $f : A \longrightarrow A$ which is surjective and **not** injective, like the one you were asked to define in the previous task. This question is about a property of A (the domain and codomain of f) that implies that such a function exists, and which is also implied by the existence of such a function. (You do not need to prove this here.)

A surjective and **not** injective function $f : A \longrightarrow A$ exists if and only if

A is infinite.

(this must be a statement about the set A, and cannot involve f)

If you want to get a deeper understanding of this point, try to prove it. You can do this in two steps: (1) You show that if A is infinite, a function exists on it that is surjective but not injective. This you can show by taking our definition of an infinite set (one that is equivalent to a proper subset of itself), and use that definition to construct such a function.

(2) Now you need to show that if such a function exists, then A is infinite. Being infinite means that A must be equinumerous to a proper subset of itself. So given a function that is surjective but not injective, you need to find a proper subset of A that is the same size as A.

If you find this confusing, have a look at the solution above, and try to figure out what a suitable proper subset of A would be, and how it is related to f.

5 [10p]

Recall that a *directed graph* (V, E) is defined as a finite set V of vertices and a relation $E \subseteq V \times V$ between them.

This question is about the properties of that relation. In the table below, make one mark in each row for the property in the left column, depending on whether all, some, or no relations defining a graph have that property. Put the mark in the corresponding ALL box, if **all relations** defining a graph have the corresponding property, the NONE box, if **no relation** has it, and the SOME box if at least one relation does, and at least one does not.

	ALL	SOME	NONE
reflexive over V		X	
transitive		X	
symmetric		X	
antisymmetric		X	
asymmetric		X	

Graphs in our definition make no special assumptions about the relations that define them, so any (finite) relation could be a graph.

6 [10p]

Recall that a *rooted tree* is a graph (T, R) such that, if the set T of nodes is not empty, then there is a node $a \in T$ (the root) such that for every $x \in T$ with $x \neq a$ there is exactly one path from a to x. Like V in the previous question, $R \in T \times T$ is a relation on the set of nodes. To make things simpler, for this question we only consider non-empty trees, that is $T \neq \emptyset$.

This question is about the properties of the relations defining trees. In the table below, make one mark in each row for the corresponding property in the left column, depending on whether all, some, or no relations defining a tree have that property. Put the mark in the corresponding ALL box, if **all relations** defining a tree have the corresponding property, the NONE box, if **no relation** has it, and the SOME box if at least one relation does, and at least one does not.

	ALL	SOME	NONE
reflexive over T			X
transitive		X	
symmetric		X	
antisymmetric	X		
asymmetric	X		

The situation is different for trees, which are much more specialized and constrained structures than graphs. Since we only consider non-empty trees, none of them can be reflexive. (If we allowed the empty tree, then its link-relation R would also be empty, which is reflexive over the empty set.)

But there are trees whose link relation is transitive, viz. all those of link height 0 or 1. (Make sure you understand why that is.) And there is a tree whose link relation is symmetric, namely the tree consisting of only a root, whose link relation is therefore empty, which is symmetric.

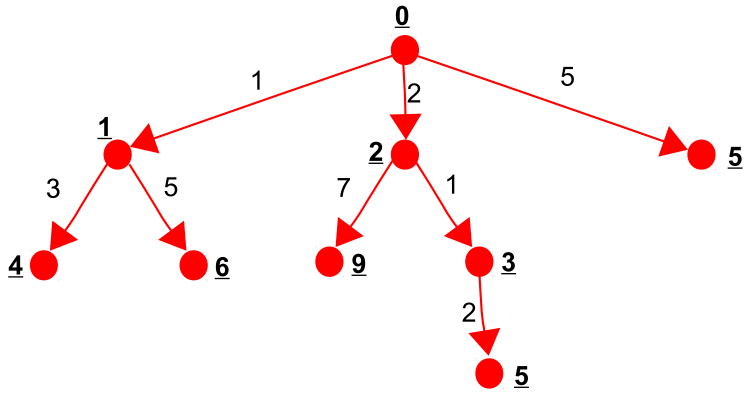
7 **[16p]**

Suppose we have a rooted tree (T, R) with nodes T, links $R \subseteq T \times T$.

We are also given a function $w : R \longrightarrow \mathbb{N}$ that assigns each *link* a natural number, let's call it the "weight" of that link.

1. [8p] Define a function $W : T \longrightarrow \mathbb{N}$ that maps each node in the tree to its "path weight", that is the sum of the weights of the links on the path from the root to that node.

<u>Example:</u>



The numbers next to the links are those assigned to them by the (given) function $w : R \longrightarrow \mathbb{N}$. The underlined numbers next to the nodes are those that your function $W : T \longrightarrow \mathbb{N}$ is supposed to compute in the case of this example.

$$W : n \mapsto \begin{cases} 0 & \text{if } R^{-1}(n) = \emptyset \\ w(m, n) + W(m) & \text{if } R^{-1}(n) = \{m\} \end{cases}$$

Since we are working with a tree, the image of every node under the inverse link relation is either empty (if the node is the root) or a singleton set (its parent). Once you have those conditions, the recursive call "walks upward" in the tree, computing the path weight of the parent, and adds the weight of the link from the parent to this node. Of course, the path weight of the root is 0.

2. [4p] Formally (using math, not natural language) define the set $L \subseteq T$ of *leaf nodes* of a tree, i.e. nodes that do not have any children:

$$L = \{n \in T : R(n) = \emptyset\}$$

A node without children is one whose image under the link relation is the empty set. Note that this is the dual of the root, which is a node that has no parent, and whose image under the *inverse* link relation is empty.

3. [4p] Formally (again, using math, not natural language) define the set $M \subseteq L$ of leaf nodes in T with the smallest path weight (you may, indeed should, use L from the previous subtask):

$$M = \{n \in L : W(n) = \min\{W(m) : m \in L\}\}$$

Once you know the set of leaves, you can compute their smallest path weight as $\min\{W(m) : m \in L\}$, and then the set of all leaves with that path weight is, well, the set of all leaves whose path weight is that number.

8 [20 p]

Find a DNF for each of the following formulae. Write “none” if a formula has no DNF.

1. [5 p] $\neg((r \barwedge \neg q) \leftrightarrow (p \vee q))$

   $(\neg p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg q \wedge r)$

```
(p q r)
(0 0 0)  -->  1
(0 0 1)  -->  0
(0 1 0)  -->  0
(0 1 1)  -->  0
(1 0 0)  -->  0
(1 0 1)  -->  1
(1 1 0)  -->  0
(1 1 1)  -->  0
```

2. [5 p] $\neg((p \barwedge q) \barwedge (r \barwedge s))$

   $(\neg q \wedge \neg s) \vee (\neg q \wedge \neg r) \vee (\neg p \wedge \neg s) \vee (\neg p \wedge \neg r)$

```
(p q r s)
(0 0 0 0)  -->  1
(0 0 0 1)  -->  1
(0 0 1 0)  -->  1
(0 0 1 1)  -->  0
(0 1 0 0)  -->  1
(0 1 0 1)  -->  1
(0 1 1 0)  -->  1
(0 1 1 1)  -->  0
(1 0 0 0)  -->  1
(1 0 0 1)  -->  1
(1 0 1 0)  -->  1
(1 0 1 1)  -->  0
(1 1 0 0)  -->  0
(1 1 0 1)  -->  0
(1 1 1 0)  -->  0
(1 1 1 1)  -->  0
```

3. [5 p] $\neg((p \,\overline{\wedge}\, q) \rightarrow (q \,\overline{\wedge}\, r))$

   $\neg p \wedge q \wedge r$

```
(p q r)
(0 0 0)  -->  0
(0 0 1)  -->  0
(0 1 0)  -->  0
(0 1 1)  -->  1
(1 0 0)  -->  0
(1 0 1)  -->  0
(1 1 0)  -->  0
(1 1 1)  -->  0
```

4. [5 p] $((p \rightarrow q) \,\overline{\wedge}\, (q \rightarrow r)) \,\overline{\wedge}\, (r \rightarrow p)$

   $(\neg p \wedge r) \vee (q \wedge r) \vee (\neg p \wedge \neg q)$

```
(p q r)
(0 0 0)  -->  1
(0 0 1)  -->  1
(0 1 0)  -->  0
(0 1 1)  -->  1
(1 0 0)  -->  0
(1 0 1)  -->  0
(1 1 0)  -->  0
(1 1 1)  -->  1
```

Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e. including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k(k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- $\overline{R}$ of a relation R: its *complement*
- R^{-1} of a relation R: its *inverse*
- $R \circ S, f \circ g$ of relations and functions: their *composition*
- $R[X], f[X]$ *closure* of a set X under a relation R, a set of relations R, or a function f
- $[a,b],]a,b[,]a,b], [a,b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are *equinumerous*
- A^* for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, ε
- $\sum S$ sum of all elements of S
- $\prod S$ product of all elements of S
- $\bigcup S$ union of all elements in S
- $\bigcap S$ intersection of all elements in S
- $\bigcup_{a \in S} E(a)$, $\bigcap_{a \in S} E(a)$ generalized union / intersection of the sets computed for every a in S