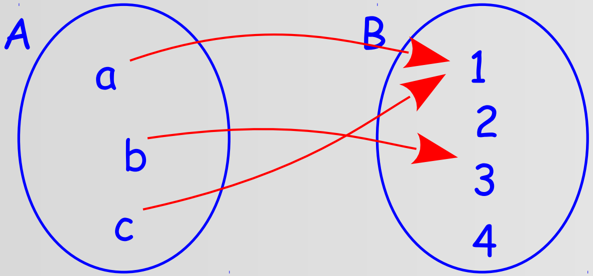
# EDAA40

# Discrete Structures in Computer Science



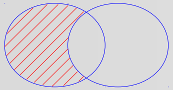

## 3: Functions

$$f : A \longrightarrow B$$

Jörn W. Janneck, Dept. of Computer Science, Lund University

$R = \{x : x \notin x\}$

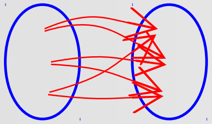
sets



$\supset$

$\heartsuit \subseteq P \times Q$

relations



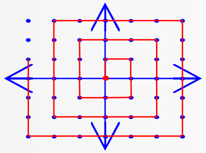
$\supset$

$f : A \longrightarrow B$

functions

$A \hookrightarrow B$

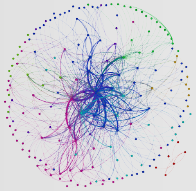
investigate



infinity

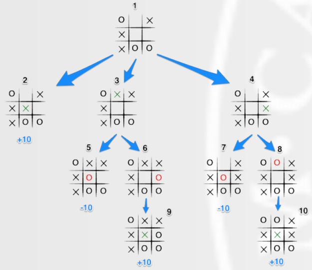
$\cup$

graphs



$\supset$

trees



working with infinite
(or arbitrarily large) stuff

definition, construction,
recursion, induction
(also: proofs, logic)

# introduction

name

parameter/argument

(name for what you feed into the function)

A function: $f(x) = x^2$

rule/expression

(how you compute the output)

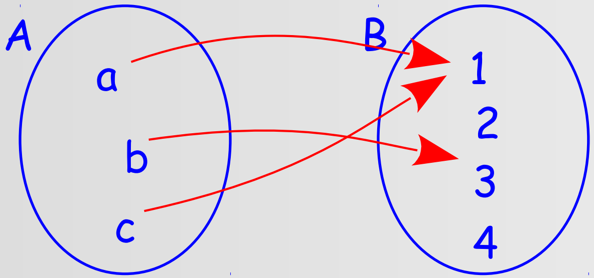
# functions are special relations

A relation $f \subseteq A \times B$ is a *function* iff

$$\mathrm{dom}(f) = A$$

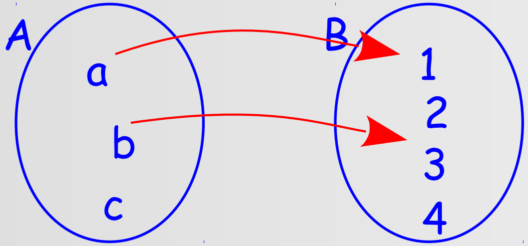
$$\#(f(a)) = 1 \text{ for all } a \in A$$

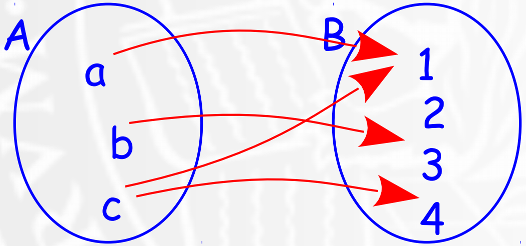
We then also write $f : A \longrightarrow B$




| f | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| a | 1 | 0 | 0 | 0 |
| b | 0 | 0 | 1 | 0 |
| c | 1 | 0 | 0 | 0 |

These aren't functions:



# domain, range, codomain

When talking about functions, some of the terminology is the same as for relations in general, some is not:

| | $R \subseteq A \times B$ | $f : A \longrightarrow B$ |
|---|---|---|
| actual values, left | domain | domain |
| actual values, right | range | range |
| A | source | domain |
| B | target | codomain |

$\mathrm{dom}(f) = A$

# set of functions

The *set of all functions from A to B* is written as

$$\langle A \longrightarrow B \rangle \qquad\qquad B^A$$

So these all say the same thing:

$$f : A \longrightarrow B \qquad\qquad f \in \langle A \longrightarrow B \rangle \qquad\qquad f \in B^A$$

# about $B^A$ ...

Why $B^A$ ?

| f | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| a | 1 | 0 | 0 | 0 |
| b | 0 | 0 | 1 | 0 |
| c | 1 | 0 | 0 | 0 |

each time, we chose from these #(B) options

we make a choice
for each of these, #(A) times

$$\underbrace{\#(B) \cdot \ldots \cdot \#(B)}_{\#(A)\ \text{times}} = \#(B)^{\#(A)}$$

So for the number of functions from A to B, we have $\#(B^A) = \#(B)^{\#(A)}$ .

How many *relations* $R \subseteq A \times B$ ?

# describing the actual *mapping*

In addition to domain and codomain, we also need to describe the actual mapping defining the function. We use this arrow $\longmapsto$ for that purpose.

$$\begin{aligned} f : A &\longrightarrow B \\ x &\longmapsto \text{(something with x)} \end{aligned}$$

We can do without the name:

$$\begin{aligned} A &\longrightarrow B \\ x &\longmapsto \text{(something with x)} \end{aligned}$$

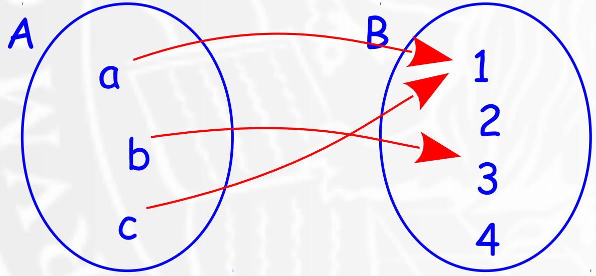
If domain and codomain are understood:

$$x \longmapsto \text{(something with x)}$$

Examples:

$$\begin{aligned} \mathrm{sqr} :\ \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned}$$

$$\begin{aligned} f :\ A &\longrightarrow B \\ v &\longmapsto \begin{cases} 1 & \text{if } v = a \text{ or } v = c \\ 3 & \text{if } v = b \end{cases} \end{aligned}$$



# functions of multiple arguments

Functions of multiple arguments (*2, 3, …, n-place functions*) are simply functions of Cartesian products:

$$\begin{aligned} \text{add} :\ & \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ & (x, y) \longmapsto x + y \end{aligned}$$

To reduce notational noise, we won't be fussy about parentheses:

$$\begin{aligned} \text{add} :\ & \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\ & x, y \longmapsto x + y \end{aligned}$$

This especially applies to *using* a function. Instead of writing

$$\text{add}((5,\ 7))$$

… we just go for

$$\text{add}(5,\ 7)$$

# restriction

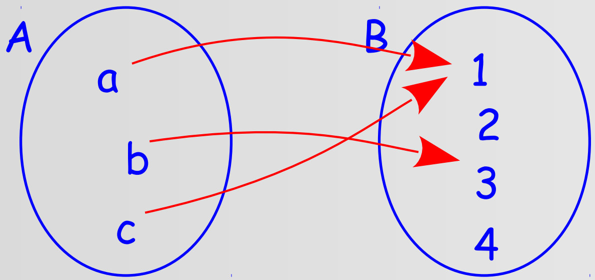
Given a function $f : A \longrightarrow B$ , its *restriction* to a set $X \subseteq A$ is defined as

$$f_X : \; X \longrightarrow B$$
$$a \longmapsto f(a)$$

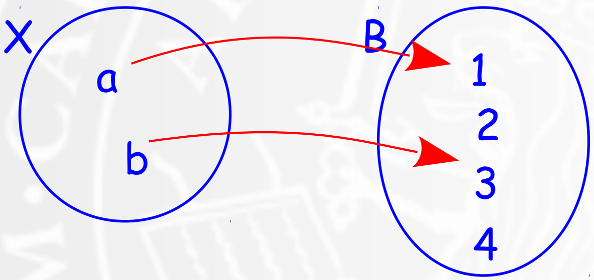
$f\,|_X$

alternative syntax

$f : A \longrightarrow B$




$f_X : X \longrightarrow B$

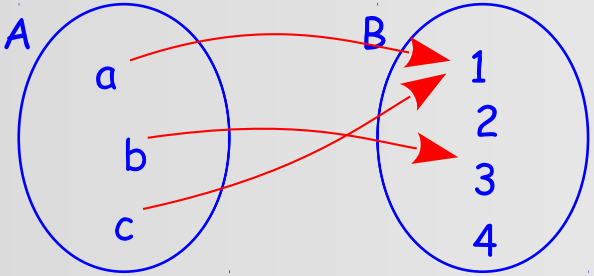


# image

Given a function $f : A \longrightarrow B$ and a set $X \subseteq A$ ,
the *image of X under f* is defined as

$$f(X) = \{f(a) : a \in X\}$$

$$f : A \longrightarrow B$$




$f(\{a, c\})$ ?

$f(\{a\})$ ?

$f(A)$ ?

# closure

An *endofunction* is one whose domain and codomain are the same set: $f : A \longrightarrow A$

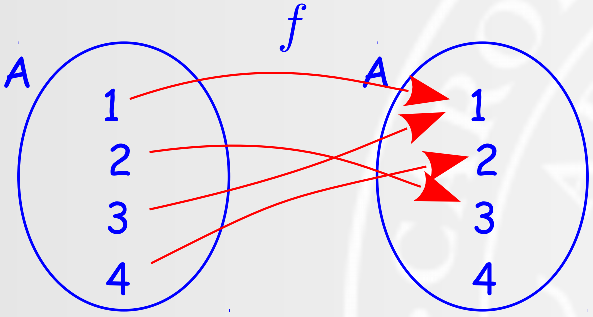
Given an endofunction $f : A \longrightarrow A$ and a set $X \subseteq A$,
the *closure of X under f* $f[X]$ is defined as the smallest $Y \subseteq A$
such that $X \subseteq Y \text{ and } f(Y) \subseteq Y$

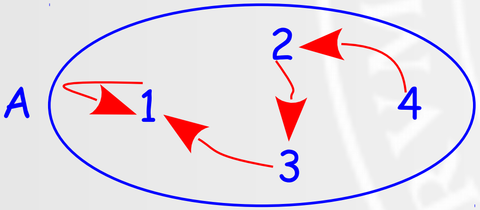
**Construction:**
(compare transitive closure)

$$Y_0 = X$$

$$Y_{n+1} = Y_n \cup f(Y_n)$$

$$f[X] = \bigcup_{i \in \mathbb{N}} Y_i$$




$f[\{1\}]$ ?
$f(\{2\})$ ?
$f[\{2\}]$ ?
$f[\{4\}]$ ?

# closure (cont’d)

$$\begin{aligned} \mathrm{incr} : \mathbb{R} &\longrightarrow \mathbb{R} \\ r &\longmapsto r+1 \end{aligned}$$

$\mathrm{incr}[\{0\}]$ ?

$\mathrm{incr}[\{r \in \mathbb{R} : r < 0\}]$ ?

$\mathrm{sqr}[[0, 0.5]]$ ?
$\mathrm{sqr}[[0, 1]]$ ?
$\mathrm{sqr}[[0, 1.1]]$ ?
$\mathrm{sqr}[[0.9, 1.1]]$ ?

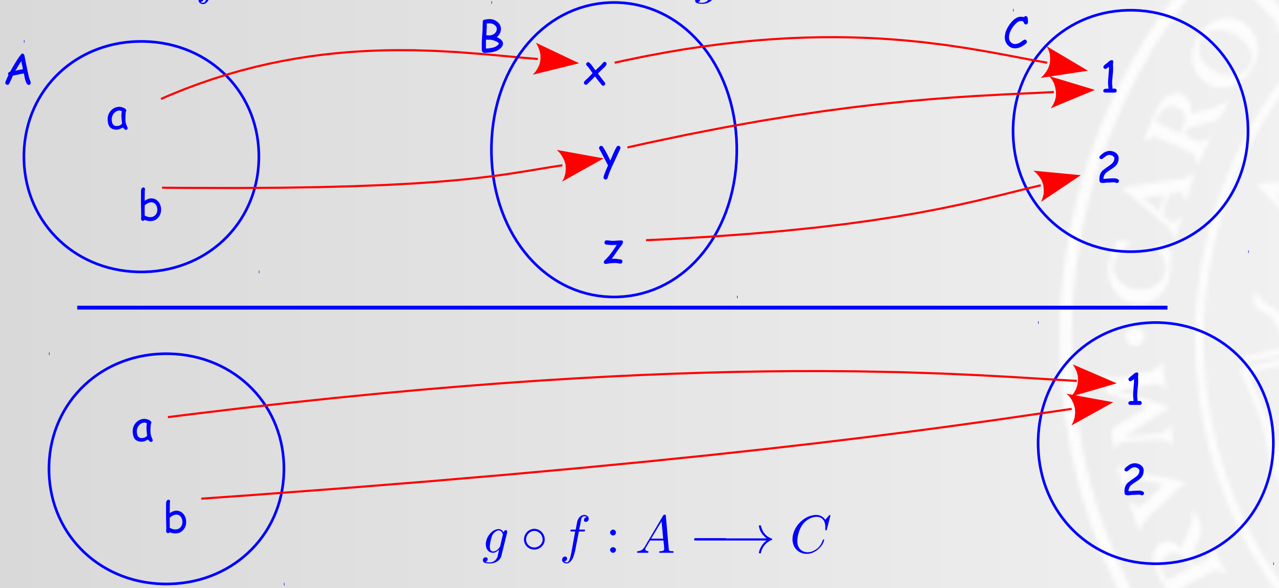
# composition

Given functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ their *composition* $g \circ f : A \longrightarrow C$ defined as:

$$g \circ f(a) = g(f(a))$$

Function composition is just a special case of composition of relations:

$f : A \longrightarrow B$ $\qquad$ $g : B \longrightarrow C$




It is associative:

$$(h \circ g) \circ f = h \circ (g \circ f)$$

So we can omit the parentheses and write

$$h \circ g \circ f$$

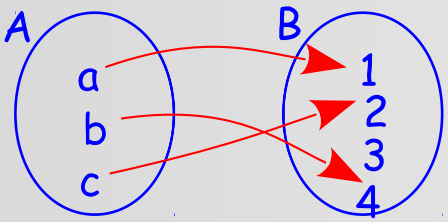
Show this.

# injection, surjection, bijection

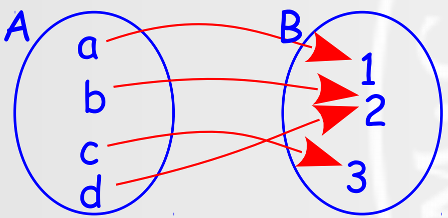
A function $f : A \longrightarrow B$ is *injective* (and thus an *injection*) iff
$a \neq b$ implies $f(a) \neq f(b)$
Notation: $f : A \hookrightarrow B$

A function $f : A \longrightarrow B$ is *surjective* (and thus an *surjection*) iff
$f(A) = B$
Notation: $f : A \twoheadrightarrow B$

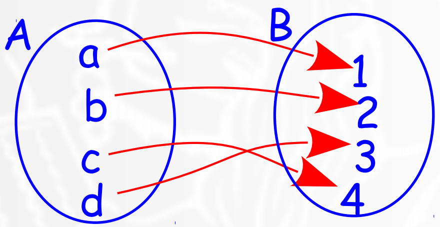
A function $f : A \longrightarrow B$ is *bijective* (and thus an *bijection*) iff
it is both injective and surjective.
Notation: $f : A \longleftrightarrow B$




injection, one-to-one
$A \hookrightarrow B$




surjection, onto
$A \twoheadrightarrow B$




bijection
$A \longleftrightarrow B$

# injection, surjection, bijection

injective ? surjective? bijective?

$$\mathrm{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$$

$$\mathrm{sqr} : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto x^2 \end{array} \qquad \mathrm{sqr}_1 : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R}^+ \\ x \longmapsto x^2 \end{array} \qquad \mathrm{sqr}_2 : \begin{array}{l} \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \\ x \longmapsto x^2 \end{array}$$

$$f : \begin{array}{l} A \longrightarrow B \\ v \longmapsto \begin{cases} 1 & \text{if } v = a \text{ or } v = c \\ 3 & \text{if } v = b \end{cases} \end{array}$$

$$A = \{a, b, c\}$$
$$B = \{1, 2, 3, 4\}$$

$$\mathrm{incr} : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ r \longmapsto r + 1 \end{array} \qquad \mathrm{incr} : \begin{array}{l} \mathbb{N} \longrightarrow \mathbb{N} \\ r \longmapsto r + 1 \end{array}$$

# inverse

Given function $f : A \longrightarrow B$ its *inverse (converse)* $f^{-1} \subseteq B \times A$
is defined as: $$f^{-1} = \{(f(a), a) : a \in A\}$$

Nothing new here – this is just a rephrasing of the definition of converses of relations in "function speak".
The term *converse* is traditionally applied to relations, *inverse* to functions.
There is no mathematical distinction between the two.

In general, the inverse of a function is not a function.

Why not?

When would it be a function?

# comparing cardinals, equinumerosity

One way to use functions is to compare the cardinalities of sets.
This is especially important for infinite sets.

For any two sets A and B, if there is an injection $f : A \hookrightarrow B$ then

$$\#(\mathrm{A}) \leq \#(B)$$

This might feel like it's just the other way: if B is at least as big as A, then there is an injection. In reality, we are **defining** the order relation on cardinalities. We'll come back to this in the next lecture.

For any two sets A and B, if there is a bijection $f : A \longleftrightarrow B$ then

$$\#(\mathrm{A}) = \#(\mathrm{B})$$

Sets with the same cardinality are called *equinumerous* (aka of the same size).

If sets A and B are equinumerous, we write $A \sim B$

# Cantor-Schröder-Bernstein theorem (CSB)

When working with infinite stuff, this theorem makes our lives a lot easier.

For any two sets A and B, if there are two injections

$$f : A \hookrightarrow B \quad \text{and} \quad g : B \hookrightarrow A$$

then there exists a bijection

$$h : A \longleftrightarrow B$$

Corollary:

If $\#(A) \leq \#(B)$ and $\#(B) \leq \#(A)$ then $\#(A) = \#(B)$ .

(Proof is a little tricky, we will omit it here. See course page for refs.)

# Cantor-Schröder-Bernstein theorem

Why does CSB require a proof? Didn't we know this already?/Isn't it obvious?

What property does this establish for $\leq$ on cardinal numbers?

Make sure you clearly distinguish between what is defined, and what needs to be proven.

"It's not what you know, but what you can prove."
Det. Alonzo Harris, LAPD

Note:

The theorem tells us *that there is* a bijection.
It does *not* tell us, what it looks like!
In other words, it is *non-constructive*.