Lecture 5: In-class exercises

Direct Proof

1.

Show that for any integer x, if x is even, then x^2 is even.

Since x is even, there exists an integer a such that $x = 2a$. Therefore, $x^2 = 4a^2 = 2(2a^2)$. Since $2a^2$ is an integer, x^2 is even.

2.

For any two integers a, b, we say that a divides b, and write $a|b$, iff there is an integer k such that $ak = b$.

Show that for any three integers a, b, c, if $a|b$ and $b|c$, then $a|c$.

Since $a|b, b|c$ there exist integers k_1, k_2 such that $ak_1 = b$ and $bk_2 = c$. Substituting the first equation into the second, we get $ak_1k_2 = c$, and since k_1k_2 is an integer, $a|c$.

3.

Show that for any two injections $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$, their composition $g \circ f : A \longrightarrow C$ is injective.

To prove their composition injective, we need to show that for any $a_1, a_2 \in A$ s.t. $a_1 \neq a_2$, it is the case that $g \circ f(a_1) \neq g \circ f(a_2)$.

Since f is injective, we get $f(a_1) \neq f(a_2)$. Since g is injective, we get $g(f(a_1)) \neq g(f(a_2))$. By definition of composition, this means $g \circ f(a_1) \neq g \circ f(a_2)$.

Direct Proof with Cases

4.

Show that every multiple of 4 equals $1 + (-1)^n(2n - 1)$ for some $n \in \mathbb{N}$.

Hint: If k is a multiple of 4, it means there is an integer $a \in \mathbb{Z}$ such that $k = 4a$. For this proof, it helps to use the cases $a = 0$, $a > 0$, and $a < 0$.

[cf. BoP, p. 99]

Contrapositive Proof

5.

Show that for any integers $x, y \in \mathbb{Z}$, if 5 does not divide xy then 5 does not divide x, and it also does not divide y.

Hint 1: The logical negation of (A and B) is (not A **or** not B).

Hint 2: This proof is best done using cases.

[cf. BoP, p. 105]

Proof by Contradiction

6.

Show that the number $\sqrt{2}$ is irrational.

Hint 1: The opposite of a number being irrational is that it can be represented as a fraction $\frac{a}{b}$ of integers a, b. It is useful to require that the fraction be fully reduced (that's how we will produce the contradiction in this case), i.e. the two integers do not have a common divisor.

Hint 2: In particular, they cannot both be even, because that would mean that 2 is a common divisor.

[cf. BoP, p. 113]