EDAA40

Discrete Structures in Computer Science

$\forall$ 9: Quantificational logic $\exists$

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objective

You should be able to read, understand, and write quantificational logic.

logic with quantifiers (informally)

Given a logical formula α that depends on a variable x :

$\forall x(\alpha)$ represents "*for all x,* α"

$\exists x(\alpha)$ represents "*there exists an x, such that* α"

alt. notations

$\bigwedge_x \alpha$

$\bigvee_x \alpha$

"Forall" is *universal* quantification, "exists" is *existential*.

SLAM, p. 218

Examples:

$\forall x(\exists y(x \perp y))$

$\forall x \exists y(x \perp y)$

$\forall x \exists y(Cxy)$

	English	Symbols
1	All composer are poets	$\forall x(Cx \to Px)$
2	Some composers are poets	$\exists x(Cx \wedge Px)$
3	No poets are composers	$\forall x(Px \to \neg Cx)$
4	Everybody loves someone	$\forall x \exists y(Lxy)$
5	There is someone who is loved by everyone	$\exists y \forall x(Lxy)$
6	There is a prime number less than 5	$\exists x(Px \wedge (x < 5))$
7	Behind every successful man stands an ambitious woman	$\forall x[(Mx \wedge Sx) \to \exists y(Wy \wedge Ay \wedge Byx)]$
8	No man is older than his father	$\neg \exists x(Oxf(x))$
9	The successors of distinct integers are distinct	$\forall x \forall y[\neg(x \equiv y) \to \neg(s(x) \equiv s(y))]$
10	The English should be familiar from Chap. 4!	$[P0 \wedge \forall x\{Px \to Px+1\}] \to \forall x(Px)$

quantifying *over a set*

In practice, we are usually interested in speaking about elements of some set.
In such cases, the set is often specified when the variable is introduced:

$\forall x \in A\ (\alpha)$ represents "*for all x in A,* α"

$\exists x \in A\ (\alpha)$ represents "*there exists an x in A, such that* α"

This is just syntactic sugar:

$\forall x \in A\ (\alpha) \dashv\vdash \forall x(x \in A \to \alpha)$

$\exists x \in A\ (\alpha) \dashv\vdash \exists x(x \in A \wedge \alpha)$

True or false?

$\forall r \in \mathbb{R}\ (\exists n \in \mathbb{N}\ (n = r))$

$\exists n \in \mathbb{N}\ (\forall r \in \mathbb{R}\ (n = r))$

$\forall n \in \mathbb{N}\ (\exists r \in \mathbb{R}\ (n = r))$

$k|n \leftrightarrow$ ... $\exists a \in \mathbb{Z}\ (ak = n)$

$n \in \mathbb{P} \leftrightarrow$ ... $n \in \mathbb{N}_2 \wedge \forall k \in \mathbb{N}_2\ (k|n \to k = n)$

$\mathbb{N}_2 = \{n \in \mathbb{N} : n > 1\}$

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more syntactic sugar

Often, one quantifier is used to introduce several variables:

$\forall x, y, z \in A\ (\alpha)$ represents "*forall x, y, z in A,* α"

$\exists x, y, z \in A\ (\alpha)$ represents "*there exist x, y, z in A, such that* α"

This, too, is just syntactic sugar:

$\forall x, y, z \in A\ (\alpha) \dashv\vdash \forall x \in A\ (\forall y \in A\ (\forall z \in A\ (\alpha)))$

$\exists x, y, z \in A\ (\alpha) \dashv\vdash \exists x \in A\ (\exists y \in A\ (\exists z \in A\ (\alpha)))$

True or false?

$\forall n \in \mathbb{N}\ (\exists a, b \in \mathbb{N}\ (a < n < b))$

$\forall a, b \in \mathbb{N}\ (\exists n \in \mathbb{N}\ (a < n < b))$

$\forall a, b \in \mathbb{N}\ (a < b \to \exists n \in \mathbb{N}\ (a < n < b))$

How can we "fix" this?

$R \subseteq A \times A$ transitive $\leftrightarrow$ ...

$\forall a, b, c \in A\ (aRb \wedge bRc \to aRc)$

$\forall a, b \in A\ (aRb \to R(b) \subseteq R(a))$

$\forall (a, b) \in R\ (R(b) \subseteq R(a))$

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the language of quantificational logic

Broad category	Specific items	Signs used	Purpose
Basic terms	Constants	$a, b, c, \ldots$	Name-specific objects: for example, 5, Charlie Chaplin, London
	Variables	$x, y, z, \ldots$	Range over specific objects, combine with quantifiers to express generality
Function letters	2-Place also 1-place!	$f, g, h, \ldots$ operators, too: +, - *, /, ...	Form compound terms out of simpler terms, starting from the basic ones
	n-Place		
Predicates	1-Place	$P, Q, R, \ldots$	For example, is prime, is funny, is polluted
also other relation symbols	2-Place		For example, is smaller than, resembles
	n-Place		For example, lies between (3-place)
	Special relation sign	$\equiv$	Identity
Quantifiers	Universal	$\forall$	For all
	Existential	$\exists$	There is
Connectives	'Not' etc.	$\neg, \wedge, \vee, \to$	Usual truth tables
Auxiliary	Parentheses and commas		To ensure unique decomposition and make formulae easier to read

terms

formulae

$\forall a, b \in A\ (aRb \to R(b) \subseteq R(a))$

$\forall k \in \mathbb{N}_2\ (k|n \to k = n)$

$\exists a \in \mathbb{Z}\ (ak = n)$

SLAM, p. 219

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more examples...

Zermelo-Fraenkel Set Theory w/Choice (ZFC)

extensionality	$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$.
regularity	$\forall x [\exists a (a \in x) \Rightarrow \exists y (y \in x \land \neg \exists z (z \in y \land z \in x))]$.
specification	$\forall w_1, \ldots, w_n \, \forall A \, \exists B \, \forall x \, (x \in B \Leftrightarrow [x \in A \land \varphi(x, w_1, \ldots, w_n, A)])$
union	$\forall \mathcal{F} \, \exists A \, \forall Y \, \forall x [(x \in Y \land Y \in \mathcal{F}) \Rightarrow x \in A]$.
replacement	$\forall A \forall w_1 \forall w_2 \ldots \forall w_n \big[\forall x (x \in A \Rightarrow \exists! y \, \phi) \Rightarrow \exists B \; \forall x \big(x \in A \Rightarrow \exists y (y \in B \land \phi)\big)\big]$.
infinity	$\exists X \, [\varnothing \in X \land \forall y (y \in X \Rightarrow S(y) \in X)]$.
power set	$\forall x \exists y \forall z [z \subseteq x \Rightarrow z \in y]$.
choice	$\forall X \left[\emptyset \notin X \implies \exists f \colon X \to \bigcup X \quad \forall A \in X \, (f(A) \in A)\right]$.

$S(x) = x \cup \{x\}$

finite transforms

Suppose we quantify all variables over a finite set D, and we have constant symbols $a_1, \ldots, a_n$ for each of its elements.

A *finite transform* of a universally/existentially quantified formula removes the quantifier, and instantiates the body for each element of D in a chained conjunction/disjunction.

Example: $\forall x(Px)$ becomes $Pa_1 \land \ldots \land Pa_n$

$\exists x(Px)$ becomes $Pa_1 \lor \ldots \lor Pa_n$

Do this for the following formula, until all quantifiers are gone. $\exists x(Qx \to \forall y(Pxy))$
Assume a domain with two values, with constant names a and b.

$\exists x(Qx \to (Pxa \land Pxb))$ $\quad (Qa \to (Paa \land Pab)) \lor (Qb \to (Pba \land Pbb))$

equivalences: distribution

The following equivalences hold for any formula α :

$$\forall x(\alpha \land \beta) \dashv\vdash \forall x(\alpha) \land \forall x(\beta)$$
$$\exists x(\alpha \lor \beta) \dashv\vdash \exists x(\alpha) \lor \exists x(\beta)$$

This should be easy to see if you think about what this would look like in a finite transform.

equivalences: quantifier interchange

The following equivalences hold for any formula α :

$$\forall x(\alpha) \dashv\vdash \neg\exists x(\neg\alpha)$$
$$\exists x(\alpha) \dashv\vdash \neg\forall x(\neg\alpha)$$

Remember de Morgan's laws?

$$\alpha \land \beta \dashv\vdash \neg(\neg\alpha \lor \beta)$$
$$\alpha \lor \beta \dashv\vdash \neg(\neg\alpha \land \beta)$$

quantifier scopes

variable *uses*, and the quantifier they are bound by

$$\underbrace{\forall x(\exists y\underbrace{(Rxy)}_{y})}_{x} \lor \underbrace{\exists z(Py \to \forall x\underbrace{(Rzx \land Ryx)}_{x})}_{z}$$

quantifier scopes, and the variables bound in/by them

free and bound variable occurrences

A variable occurrence is *bound* iff it occurs inside the scope of a quantifier that binds that variable.

It is *free* otherwise.

A formula with no free variable occurrences is called *closed*.
A closed formula is a *sentence*.

free occurrences

$$\forall z[Rxz \to \exists y(Rzy)]$$

bound occurrences

equivalences: vacuity, relettering

Vacuity: If x does not occur free in α , then

$$\forall x(\alpha) \dashv\vdash \alpha \dashv\vdash \exists x(\alpha)$$

This doesn't work if quantifying over an empty set.

Relettering: If x does not occur at all in α , and α' is the result of replacing every bound occurrence of some variable y in α with x, then

$$\alpha \dashv\vdash \alpha'$$

Example:

$$\forall y(Ry \to Qyz) \dashv\vdash \forall x(Rx \to (Qxz))$$

interpretations

The value of a formula depends on how you read the symbols in it.

$\forall z(Rxz \to \exists y(Rzy))$ $\forall k \in \mathbb{N}_2\ (k|n \to k = n)$ $\exists a \in \mathbb{Z}\ (ak = n)$

Also, we need to determine what values the quantified variables can assume.

A *domain* or *universe* D are the values that quantified variables range over. (For example: all sets in the case of the axioms of set theory.)

An *interpretation* v is a function assigning mathematical objects to the symbols occurring in a formula. Specifically...

- to each constant a $v(a) \in D$
- to each variable x $v(x) \in D$
- to each n-place function letter f $v(f) : D^n \longrightarrow D$
- to each n-place relation letter P $v(P) \subseteq D^n$
- to the identity symbol $\equiv$ the identity over D

evaluating terms and formulae

Given a domain D and an interpretation v, the value of a term t is defined as follows:

$$v : t \mapsto \begin{cases} v(a) & \text{for every constant } a \\ v(x) & \text{for every variable } x \\ v(f)(v(t_1), ..., v(t_n)) & \text{for } t = f(t_1, ..., t_n) \end{cases}$$

With this, we can determine the truth value (0 or 1) of a formula as follows:

$$v : \alpha \mapsto \begin{cases} v(t_1) = v(t_2) & \text{for } \alpha = t_1 \equiv t_2 \\ (v(t_1, ..., v(t_n)) \in v(P) & \text{for } \alpha = Pt_1...t_n \\ \neg v(\beta) & \text{for } \alpha = \neg\beta \\ v(\beta) \wedge v(\gamma) & \text{for } \alpha = \beta \wedge \gamma \\ ... & \\ ? & \text{for } \alpha = \forall x(\beta) \\ ? & \text{for } \alpha = \exists x(\beta) \end{cases}$$

SLAM 9.3, pp. 227-233

logical implication

Given a set of formulae $A = \{\alpha_1, ..., a_n\}$ and a formula β , we say that A *logically implies* β iff there is no interpretation v such that all the formulae in A are true under v, but β is false:

$$A \vdash \beta \iff \neg\exists v(\neg v(\beta) \wedge \forall\alpha(\alpha \in A \to v(\alpha)))$$

Also:

$\alpha \dashv\vdash \beta \iff \alpha \vdash \beta \wedge \beta \vdash \alpha$	logical equivalence
$\emptyset \vdash \alpha$	logical truth
$\emptyset \vdash \neg\alpha$	contradiction

A formula that is neither logically true nor a contradiction is *contingent*.

some implications