EDAA40 Exam

03 June 2019

Instructions

Things you CAN use during the exam.
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam.
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favorable assumption about what you intended to write.

Good luck!

1	2	3	4	5	6	PC	total
10	10	10	35	20	15	(5)	100

Total points: 100

points required for 3: 50

points required for 4: 67

points required for 5: 85

PC: points received from the programming contest

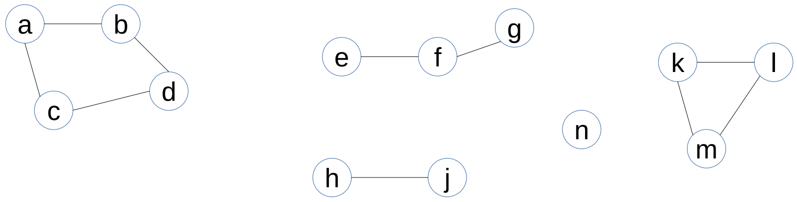

Figure 1: Example, villages and roads between them.

1 [10p]

Suppose we have a graph (V, R) with vertices V representing villages and edges $R \subseteq V \times V$ representing the roads between them, similar to the one in Figure 1 above. The road relation is *symmetric*, that is $\forall v_1, v_2 \in V \ ((v_1, v_2) \in R \Leftrightarrow (v_2, v_1) \in R)$.

1. [5 p] In the example above in Figure 1, we have the vertices
   $V = \{a, b, c, d, e, f, g, h, j, k, l, m, n\}$.
   Give the edge relation R for the example:

   $R =$

2. [5 p] Define, for any such graph (V, R) (not just for the example above!), the set D of dead-ends, i.e. the set of all villages one cannot leave (i.e. go to different village) at all by road, or by at most one road. In the example, this set would be $D = \{e, g, h, j, n\}$.

   $D =$

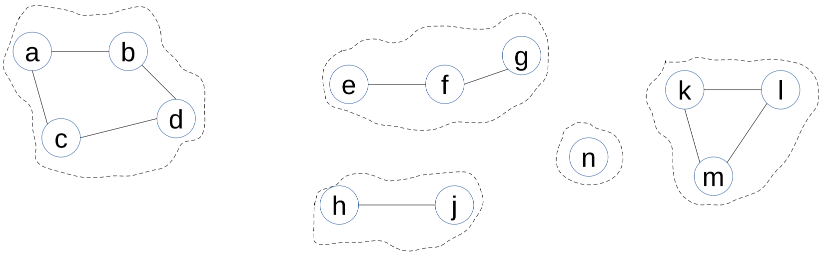

Figure 2: Grouping villages into islands

2 [10 p]

Suppose the villages and roads in (V, R) from task 1 are located on a set of islands, and each island corresponds exactly to the villages that can be reached from one another by road (i.e. all the villages on an island are connected by roads, not necessarily directly), as shown for our example in Figure 2.

Define for any such (V, R) the set A of islands, that is a set of sets of villages that can reach each other by road. In the example, we would have

$$A = \{\{a, b, c, d\}, \{e, f, g\}, \{h, j\}, \{k, l, m\}, \{n\}\}.$$

$$A =$$

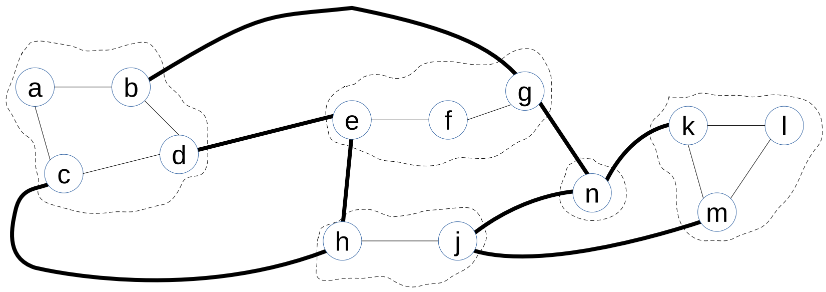

Figure 3: Our example villages and roads on the archilepago, with ferry connections between islands drawn in bolder lines.

3 [10 p]

Suppose that in addition to the villages in V and the symmetric relation of the roads connecting them R from task 1 we also have another *symmetric* relation $F \subseteq V \times V$ of ferry connections connecting villages on different islands, as in Figure 3. We assume that $F \cap R = \emptyset$, that is that there is no ferry connecting villages that are also connected by a road.

Define the set $Ports$ of ports, that is villages that have at least one ferry connection. In the example, this would be $Ports = \{b, c, d, e, g, h, j, k, m, n\}$.

$$Ports =$$

4 [35 p]

Suppose the villages V, the roads $R \subseteq V \times V$, and the ferry connections $F \subseteq V \times V$, as in task 3. The combined connections between all the villages, using roads and ferries, is the relation $E = R \cup F$. (Recall that R and F are disjoint, so $R \cap F = \emptyset$.)

We want to define a function $P : V \times V \longrightarrow \mathcal{P}(V^*)$ such that $P(v_1, v_2)$ returns all cycle-free paths from v_1 to v_2 in E, that is all paths such that every vertex (village) occurs at most once in it. If $v_1 = v_2$, it returns a set containing only the empty path from v_1 to itself, that is $P(v_1, v_1) = \{v_1\}$.

We define P using a helper function $P' : V^* \times V \times V \longrightarrow \mathcal{P}(V^*)$ that builds the path as it searches for its destination:

$$\begin{aligned} P &: V \times V \longrightarrow \mathcal{P}(V^*) \\ &v_1, v_2 \mapsto P'(\varepsilon, v_1, v_2) \end{aligned}$$

1. [25 p] Define P'. You might find it useful to talk about the set of all vertices in a sequence of vertices (that is, in a path) – if $p \in V^*$, then you can use $\operatorname{set}(p)$ to describe the set of all vertices that occur in p.

   $P' : V^* \times V \times V \longrightarrow \mathcal{P}(V^*)$

   $p, v, w \mapsto$

2. [10 p] Define the set $\Pi \subseteq V^*$ of all non-cyclic paths in (V, E), as computed by P:

   $\Pi =$

5 [20 p]

Suppose we have a function $d : E \longrightarrow \mathbb{R}^+$ from the combined connections to the positive real numbers, signifying for each $(v_1, v_2) \in E$ the time it takes to travel from v_1 to v_2 (by either road or ferry, depending on whether $(v_1, v_2) \in R$ or $(v_1, v_2) \in F$). For any $(v_1, v_2), (v_2, v_1) \in E$, it is NOT guaranteed that $d(v_1, v_2) = d(v_2, v_1)$.

Given a path $v_0 v_1 v_2 ... v_n$ of length n in E, we want to measure it regarding the total time it takes and also the number of ferry connections in the path. The function $M : \Pi \longrightarrow \mathbb{N} \times \mathbb{R}_0^+$ computes for each non-cyclic path $p \in \Pi$ (the set of all non-cyclic paths from the previous task) a pair (n, r), where n is the number of ferry connections and r the sum of the travel time of all the connections, by road or ferry, in the path.

For every $v \in V$, applying M to the empty path v results in $(0, 0)$, i.e. $\forall v \in V \ (M(v) = (0, 0))$.

Define M.

$M : \Pi \longrightarrow \mathbb{N} \times \mathbb{R}_0^+$

$p \mapsto$

Hint: you might find it useful to distinguish cases where p has the form v (i.e. it is an empty path), from cases where it has the form $v_1 v_2 q$, i.e. a path with at least two vertices (i.e. of length at least 1).

6 [15 p]

With $P : V \times V \longrightarrow \mathcal{P}(V^*)$ from task 4 we can compute the set $P(v_1, v_2)$ of all non-cyclic paths between two villages v_1 and v_2, and with $M : \Pi \longrightarrow \mathbb{N} \times \mathbb{R}_0^+$ from task 5 we can measure each path in terms of the number of ferry connections involved and its total time.

We are interested in the set of "best" paths, which we define as follows:

Suppose $M(p_1) = (n_1, r_1)$ and $M(p_2) = (n_2, r_2)$.
Path p_1 is "better" than p_2. written as $p_1 \prec p_2$, iff $(n_1 < n_2) \vee (n_1 = n_2 \wedge r_1 < r_2)$.

Using the function P defined in task 4, define the function $\hat{P} : V \times V \to \mathcal{P}(V^*)$ such that $\hat{P}(v_1, v_2)$ returns the set of best paths between v_1 and v_2.

Note:

1. $\forall v_1, v_2 \in V \; (\hat{P}(v_1, v_2) \subseteq P(v_1, v_2))$
2. $\forall v_1, v_2 \in V \; \forall p, q \in \hat{P}(v_1, v_2) \; (M(p) = M(q))$

$\hat{P} : V \times V \longrightarrow \mathcal{P}(V^*)$

$v_1, v_2 \mapsto$