# EDAA40 Exam

EDAA40-2023 Exam #1

2023-06-03

**Things you CAN use during the exam:**
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

**Things you CANNOT use during the exam:**
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, …

**WRITE CLEARLY.** If I cannot read/decipher/make sense of something you write, I will make the least favourable assumption about what you intended to write.

**A sheet with common symbols and notations and with information about grading is attached at the end.**

**Possible solution or hints:** This version includes a reference solution, marked like this paragraph. **Note:** the reference solution often offers additional explanations / proof sketches beyond what the question asked for, to help students who use it to study for future exams. Students were not required to explain their anwers unless the question explicity requested an explanation.

> This version includes **grading guidelines**, marked like this paragraph.

Good luck!

| Question: | 1 | 2 | 3 | 4 | 5 | **Sum** |
|---|---|---|---|---|---|---|
| Max Points: | 16 | 12 | 28 | 26 | 18 | 100 |
| Points Reached: | | | | | | |

| | |
|---|---|
| Total points: | 100 + lab bonus |
| Points required for **3**: | 50 |
| Points required for **4**: | 67 |
| Points required for **5**: | 85 |

# Grading Template

## Notation

- 2 foo
  - 1 quux
  - 1 blah
- 2 bar

means that there are $4$ points to distribute; 2 for "foo" and 2 for "bar". Of the 2 points for "foo", $1$ is for "quux" and $1$ is for "blah". Count incorrect answers as $0$ (but see below).

## Alternatives

- 2 Take one of the two routes
  - **Route A**:
    - 2 Did *foo*
  - **Route b**:
    - 2 Did *bar*

Some questions have alternatives. Here, first check which approach the student took, then use the appropriate scheme.

## Capping negative points

- 2 foo
  - -1 Did $X$
  - -1 Did $Y$
  - -1 Did $Z$
- 2 bar
- -3 Did everything in log space

Some questions use negative points. Negative points are capped to $0$ at their parent. So an answer that does "foo" but also did $X$, $Y$, and $Z$ gets $2 - 3 = -1$ points, but this is capped at $0$, so if "bar" is correct, the total is still $2$. However, if the answer "did everything in log space", then the $-3$ apply across "foo" and "bar", so that "did $X$" + "did everything in log space" would give zero points for the question in total.

## Cases that are not covered

Students are creative. While grading, it may turn out that something is unfair, or that a special case needs consideration. *Make a note somewhere if you deviate from the grading scheme.*

- For one-off cases, use your best judgment.
- For repeating patterns:
  - Look for a generalisable rule, write it down
  - **If you have few (ca. $\leq 20$) earlier exams to revisit**: consider doing so
  - **Otherwise (incl, you decide not to revisit)**: determine the maximum delta that your change can make, and add it to your notes (e.g., "Q3.5: -0.5 to +2"). We can later use this information to revisit only those exams where your change has a chance of affecting the grade.

Ideally, we can use a shared document to track this information, but keep in mind that there must be no personally identifiable information in such documents, so tracking it on the CS git is likely safest.

## If In Doubt, Be Generous

## Question 1 (16 Points)

Let $K \subset \mathbb{N}$ such that $K = \{k_1, k_2\}$, with $1 < k_1 < k_2$. Let $I_K \supseteq K$ the minimal superset of $K$ with the property that $a \in I_K$ and $k \in K \implies a + k \in I_K$.

(a) (4 Points) Define an *injective* function $f_K : \mathbb{N} \to I_K$:

$$f_K = n \mapsto (n+1) \cdot k_1$$

2 Maps to $I_K$ (i.e., multiples of $k_1$ plus multiples of $k_2$ only, excluding 0)
2 Injective

(b) (3 Points) Is your function surjective / bijective? Explain.

**Possible solution or hints:** No. Counter-example: Assume $k_1 = 2$, $k_2 = 3$. Then there is no $n \in \mathbb{N}$ s.th. $f_K(n) = 3$, even though $k \in I_K$.

1 Correct answer (most likely "no")
2 Explanation, e.g., counterexample

(c) (6 Points) Define a relation $R_K$ such that $I_K = R_K[K]$. Your definition of $R$ must not be recursive or circular.

$$R_K = \{\langle a, c\rangle | a \in \mathbb{N}, b \in \{k_1, k_2\}, c = a + b\}$$

1 $R_K$ contains pairs, e.g., $\langle a, b\rangle$
2 $a$ is from sufficiently large set, e.g., $\mathbb{N}$ or $I_K$
1 $b$ is sum of $a$ with something
2 The elements added to $b$ are from $\{k_1, k_2\}$

(d) (3 Points) Is your $R_K$ reflexive? Is it symmetric? Is it transitive?

**Possible solution or hints:** No to all of the above.

1 reflexive (most likely **no**)
1 symmetric (most likely **no**)
1 transitive (most likely **no**)

## Question 2 (12 Points)

Assume that $a$, $b$, and $c$ are atomic propositions in Boolean propositional logic.

(a) (4 Points) Is the propositional Boolean logic formula $a \rightarrow ((\neg b) \vee c)$ equivalent to $(b \wedge a) \rightarrow c$? Justify your statement using the techniques that we studied in the course.

**Possible solution or hints:** Several approaches possible; ANY of the approaches below is sufficient:

**Solution approach #1: Truth Tables**

| a | b | c | a → ((¬b) ∨ c) | (b ∧ a) → c |
|---|---|---|---|---|
| **F** | **F** | **F** | *T* | *T* |
| *T* | **F** | **F** | *T* | *T* |
| **F** | *T* | **F** | *T* | *T* |
| *T* | *T* | **F** | **F** | **F** |
| **F** | **F** | *T* | *T* | *T* |
| *T* | **F** | *T* | *T* | *T* |
| **F** | *T* | *T* | *T* | *T* |
| *T* | *T* | *T* | *T* | *T* |

**Solution approach #2: Transform one term to the other via tautological equivalence** Example:

$$\begin{aligned} a \rightarrow ((\neg b) \vee c) &\iff a \rightarrow (b \rightarrow c) && (\rightarrow \text{intro}) \\ &\iff (a \wedge b) \rightarrow c && (\text{uncurrying}) \\ &\iff (b \wedge a) \rightarrow c && (\wedge\text{-commutativity}) \end{aligned}$$

**Solution approach #3: Transform both to DNF or other normal form, compare** Both formulae transform into the following (first formula: by expanding definition of →; second formula: by expanding def. of → followed by applying DeMorgan equivalence):

$$(\neg a) \vee (\neg b) \vee c$$

1 Picked a suitable approach

3 Used the approach appropriately:

- **Approach 1**:
  - 2 Complete truth table (or equivalent summary)
    - · One "free" mistake at the level of a transcription error
    - -1 for misinterpreting implication semantics
    - -0.5 for individual mistakes not covered by the previous points
  - 1 Correct interpretation of results (even if table is incorrect)
- **Approaches 2/3**:
  - 2 Showed how derivation was done:
    - 1 Showed intermediate step (or final step, for #3)
    - 1 Showed additional intermediate step, or mentioned appropriate tautological equivalence for additional intermediate step
  - 0.5 Correct derivation
  - 0.5 Correct interpretation of results (even if derivation was wrong)

(b) (4 Points) Which of the following hold? For each row, mark if the statement **always** holds, **never** holds, or is **contingent**, in the choice table below:

| Statement | always | never | contingent |
|---|---|---|---|
| $\vdash b \wedge \neg b$ | | X | |
| $\vdash a \vee (b \rightarrow \neg a)$ | X | | |
| $\neg b \vdash a \vee \neg(a \rightarrow b)$ | X | | |
| $\vdash (a \rightarrow b) \leftrightarrow (b \rightarrow \neg a)$ | | X | X |
| $b \vdash (a \leftrightarrow \neg c) \vee ((\neg(c \vee b)) \rightarrow a)$ | X | | |

To undo a checkmark, circle it.

*Note on grading:* Each row (except for the example) is worth one point. Each incorrect answer counts as $-1$, while each omitted or ambiguous answer (multiple checkmarks) counts as $\pm 0$. A negative points total on this table counts as zero points total for the table.

**Possible solution or hints:** The next-to-last statement is never true, since the empty left-hand side of the $\vdash$ (equivalent to $\models$ in SLAM, as discussed in class) specifies that we know nothing about $a$ and $b$ and that it would therefore have to hold for any truth assignment for these variables for the statement itself to hold.

However, this type of question was treated differently in earlier exams, so we also allow the answer "contingent".

- One point per question
- second-to last question: either answer is fine

(c) (4 Points) Are the following formulas $\phi$ and $\psi$ equivalent for every $S$ and every $T$? Explain your answer.

$$\phi = \forall x \in S. (\exists y \in S. ((x + y \in S) \vee (x \in T)))$$
$$\psi = \exists y \in S. (\forall x \in S. ((x + y \in S) \vee (x \in T)))$$

**Possible solution or hints:** No. Counter-example: consider $S = \emptyset, S = \{-3, -1, 2, 3\}$: here, $\phi$ holds, since for $x \in \{-3, -1\}$, we can set $y = 2$, for $x = 2$, we can set $y = -3$, and for $x = 3$, we can set $y = -1$. However, $\psi$ does not hold: there is no single $y$ such that for all $x$, $x + y$ is in $S$.

1 Correct answer

3 Suitable explanation:

- 1 Answer shows understanding that $\forall\exists$ is not the same as $\exists\forall$
- 1 Answer shows understanding of *why* $\forall\exists$ is not the same as $\exists\forall$
- 1 Suitable argument (counter-example, high-level description)

## Question 3 (28 Points)

Assume that $\langle T, R\rangle$ is a nonempty directed tree with $n = \#T$ elements and root $a$, and an injective function $f : D \to T$, where $D = [0, n)_{\mathbb{N}} = \{0, \ldots, n-1\}$ is an interval in $\mathbb{N}$. $f$ has the following properties:

($P_1$) $f(0) = a$

($P_2$) For any $p, c \in D$, if $\langle f(p), f(c)\rangle \in R$, then $p < c$.

($P_3$) $f^{-1} : T \to D$ is also an injective function.

(a) (6 Points) Let $x, y \in D$. Then $f(x)$ and $f(y)$ are tree nodes. If we know that $x = 2 \cdot y$, can there be a path from $f(x)$ to $f(y)$? Must there be such a path? Explain in your own words.

**Possible solution or hints:** There can be no such path. Proof by case distinction and contradiction:

- case $x = y = 0$: Such a path would be from $a$ to $a$ (due to ($P_1$), which contradicts the requirement that in directed trees, there must be no such path.
- case $x \geq 1$: This implies that $x > y$. By ($P_2$), any edge that goes to $f(y)$ has to come from a node index $z$ such that $z < y$. Thus, for any path from $x$ to $y$, all edges $\langle l, r\rangle$ on that path would have the property that $f^{-1}(l) < f^{-1}(r)$. By transitivity of $<$, such a path can only exist if $x < y$, which contradicts $x > y$.

2 case $x = 0$

- 0.5 Aware that this is special case
- 0.5 "not possible" in this case
- 1 Utilises tree axioms or other suitable properties from class
  - =0.5 partial credit if not utilised tree axioms but explicitly utilised ($P_1$)

4 case $x > 0$

- 1 "not possible" in this case
  - =0 Arrived at wrong conclusion because of reading $x = 2 \cdot y$ as $y = 2 \cdot x$ (can still get up to 3 of 4 points for this part of the question)
- 1 Utilises ($P_2$) in **explanation**
- 2 Utilises transitivity of $<$ (no need to name explicitly) in **explanation**, i.e. does not only look at parent/child, but also ancestor/descendant
- -0.5 Mixes up elements from $D$ and from $T$ in formal notation at least twice
- -0.5 Mixes up elements from $D$ and from $T$ in all cases

(b) (8 Points) Define a recursive function $d : D \to \mathbb{N}$ such that $d(x)$ is the *depth* of $f(x)$, i.e., the length of the path from the root $a$ to $f(x)$, or 0 if there is no such path.

$$d = x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ 1 + d(f^{-1}(R^{-1}(f(x)))) & \text{if } x > 0 \end{cases}$$

2 Case distinction 0 vs non-0
2 Recursive call on the parent index
-1 Recursion uses parent node (from $T$) instead of index (from $D$)
2 Utilise $R$ directly or indirectly to obtain parent
1 Use $f^{-1}$ or equivalent
1 Use $f$ or equivalent
-3 Recursion can never terminate (e.g., recursion on self– ignore distinction between $D$ and $T$ for this)
-3 Recursion can terminate, but is trivial (e.g., jumps directly to root)
-2 Recursion can terminate, but goes the wrong way (descendants, siblings, …)
-1 Recursion can terminate, but hops over parent (e.g., grandparent)

(c) (6 Points) Consider the case where additionally, for all $p, c \in D$, $\langle f(p), f(c) \rangle \in R \implies p = \lfloor \frac{c}{2} \rfloor$. What is the maximum number of children of any node, across all nodes in such a tree? Explain your answer in your own words.

**Possible solution or hints:** Each node may have at most two children. Consider a tree with $n \geq 4$: node $f(1)$ will have $f(2)$ and $f(3)$ as children (since $\lfloor \frac{2}{2} \rfloor = 1$ and $\lfloor \frac{3}{2} \rfloor = 1$), so the maximum can be no less than 2. Now assume that we have a node $f(z)$ with 3 children. Then there must be three distinct natural numbers $C = \{c_1, c_2, c_3\}$ with the property that for $c \in C$, $z = \lfloor \frac{c}{2} \rfloor$. However, if $c < z \cdot 2$, then $\lfloor \frac{c}{2} \rfloor < z$, and if $c \geq (z+1) \cdot 2$, then $\lfloor \frac{c}{2} \rfloor > z$, so $C \subseteq \{z \cdot 2, z \cdot 2 + 1\}$, which contradicts $C$ containing three distinct elements.

- Correctly utilises the rounding function:
  - **yes**, rounds
    - 2 Show that 2 children are possible
      - 1 Mentions that this should be shown
      - 1 Suitable explanation or example
    - 4 Show that there can be no more than 2 children
      - 1 Mentions that this should be shown
        - 0.5 For $x$ children
        - 0.5 $x = 2$
      - 3 Suitable argument
        - 2 Explicitly or implicitly exploits effect of rounding
        - 1 Makes argument that greater/smaller elements must always have higher/lower parent (at least one of these)
  - **no**, does not round (max 2 points)
  - 2 Highlights that anything nontrivial is not a tree, with a suitable argument or example

(d) (4 Points) For our directed tree $\langle T, R \rangle$ with root $a$, let $\{a\} \subseteq T' \subseteq T$, and let $R' = (T' \times T') \cap R$. Is $\langle T', R' \rangle$ always a tree, sometimes a tree, or never a tree? Explain in your own words.

**Possible solution or hints:** It is **sometimes** a tree:

- Set $\langle T', R' \rangle = \langle T, R \rangle$. Then $\langle T', R' \rangle$ is trivially a tree.
- Assume that $T = \{a, x, y\}$ and $R = \{\langle a, x \rangle, \langle x, y \rangle\}$, and let $T' = \{a, y\}$. Then $R' = \emptyset$. In this graph $\langle T', R' \rangle$, there is no path from $a$ to $y$, so it cannot be a tree.

1 Correct answer
1 Example of "is a tree" case or suitable argument
2 Example of "is not a tree" case or suitable argument

(e) (4 Points) Let $S = R \circ R^{-1}$. Which of the following properties hold for $S$? Mark exactly *one* field per row (**always**, **never**, **sometimes**) in the table below:

| Property | always | never | sometimes |
|---|---|---|---|
| **Reflexive** | | X | |
| **Symmetric** | X | | |
| **Antisymmetric** | | | X |
| **Transitive** | X | | |

To undo a checkmark, circle it.

*Note on grading:* Each row is worth one point. Each incorrect answer counts as $-1$, while each omitted or ambiguous answer (multiple checkmarks) counts as $\pm 0$. A negative points total on this table counts as zero points total for the table.

**Possible solution or hints:** Explanations: $S$ represents a "sibling-or-self" relationship (except for $a$), i.e., "both nodes have a parent, and that parent is identical".

- $S$ cannot be **reflexive**, because $aSa$ never holds.
- $S$ is always **symmetric** (to prove this formally, try showing e.g. that $S = S^{-1}$)
- $S$ is **antisymmetric** for trees in which each node has at most one child, but not in trees in which any node has more than one child.
- $S$ is always **transitive** (to prove this formally, try showing that $aSb$ iff there exists a node $p$ such that $pRa$ and $pRb$ and use the uniqueness of the parent for each node)

- One point per entry

# Question 4 (26 Points)

In a certain game, a set of players $P$ are competing to *connect* different train stations via fields that the players can buy. We can think of the fields as vertices on a directed graph $\langle V, E\rangle$, with the train stations $S \subset V$, where the edges $E$ are directed connections between two fields. Only one player $p \in P$ at a time may *own* a field, and we write $v \in O_p$ if and only if player $p$ owns vertex $v$.

Analogously to *paths*, assume that a *route* from $v_a, v_b \in V$ is a sequence of vertices $v_0 \ldots v_k$ with $k \in \mathbb{N}$, where $v_0 = v_a$ and $v_k = v_b$ and $\langle v_i, v_{i+1}\rangle \in E$ for all $i \in \mathbb{N}, i < k$. For a given route, we call the set $\{v_i \mid i \in \mathbb{N}.i \leq k\}$ the *vertices on the route*.

A player $p$ *is connecting* vertex $v_s \in V$ to vertex $v_e \in V$ if there is a route from $v_s$ to $v_e$, and all vertices on that route are from $O_p \cup \{v_s, v_e\}$. Thus, a player can connect two train stations even if another player owns the stations themselves.

(a) (14 Points) Complete the definitions below:

- Define the set of *free vertices* $V_0 \subseteq V$ that are not owned by any player.

$$V_0 = V \setminus \bigcup_{p \in P} O_p$$

General: effective use of notation is key to this exercise. Interpret notational mistakes as wro answers if they introduce ambiguity.

3 Subtotal

1 Include all relevant vertices
2 Eliminate all player-owned vertices

- Define the relation $R_p \subseteq V \times V$ of all *player routes* for player $p$, such that $\langle v_1, v_2\rangle \in R_p$ if and only if there is a route from $v_1$ to $v_2$ where all vertices on the route are in $O_p$.

$$R_p = (E \cap (O_p \times O_p))^*$$

**Possible solution or hints:** Where $S^*$ is the reflexive and transitive closure of the binary relation $S$. Alternatively, one can construct the transitive closure, $(E \cap (O_p \times O_p))^+$ and extend it with the identity relation $\{\langle v, v\rangle | v \in V\}$.

3 Subtotal

1 Uses transitive or reflexive+transitive closure
$\pm 0$ Ignore omission of reflexive closure, since it is reasonable in this application t only be interested in routes of lengths greater than zero.
1 Intersection that involves $E$
1 Work with $O_p \times O_p$
-1 No attempt to construct transitivity

- Define $T_p \subseteq S \times S$ such that $s_1 T_p s_2$ if and only if player $p$ is connecting $s_1$ to $s_2$.

$$\begin{aligned} T_p &= (E \circ R_P \circ E) \cap (S \times S) \\ &= \{\langle s_1, s_2\rangle \mid s_1, s_2 \in S \wedge \exists v_1, v_2 \in V. \\ &\quad (s_1 E v_1 \wedge v_1 R_p v_2 \wedge v_2 E s_2)\} \end{aligned}$$

**Possible solution or hints:** Either of the above definitions (or an equivalent definition) suffices.

4 Subtotal

- 1 Use $R_p$ or equivalent
- 1 Prepend $E$ or equivalent
- 1 Append $E$ or equivalent
- 1 Draw from/filter by $S \times S$
  - -0.5 $T_p$ includes connections that are not from $S \times S$

- Give a formula that holds if and only if a player is connecting all stations in the game to all other stations, in either direction (i.e., connecting $s_1$ to $s_2$ is sufficient, there is no need to also connect $s_2$ to $s_1$).

**Possible solution or hints:**

$$\exists p \in P.(\forall s_1, s_2 \in S.((s_1 \neq s_1) \rightarrow ((s_1 T_p s_2) \vee (s_2 T_p s_1))))$$

All of the parentheses in the above formula are optional; using standard precedence, we can write

$$\exists p \in P.\forall s_1, s_2 \in S.s_1 \neq s_1 \rightarrow s_1 T_p s_2 \vee s_2 T_p s_1$$

4 Subtotal

- 1 exists a player
- 1 check all nonequal pairs of stations
  - -0.5 Also checks paths from same station to itself
- 1 check for connecting in both directions
  - -0.5 require both instead of requiring either
- 1 use $T_p$ or equivalent effectively

(b) (12 Points) Recursively define the set of potential player routes $C_p : \mathbb{N} \rightarrow \mathcal{P}(V \times V)$ that player $p$ could build, thus extending $R_p$, by buying up to $n$ additional vertices. Assume that the only new vertices that the player can buy are from $V_0$. Explain in your own words why your definition is well-defined (i.e., why it does not get "stuck" in a cycle during recursion).

$$C_p = n \mapsto \begin{cases} R_p & \text{if} \quad n = 0 \\ (E_{p,0} \circ C_p(n-1)) \cup (C_p(n-1) \circ E_{p,0}) & \text{if} \quad n \geq 1 \end{cases}$$

where

$$E_{p,0} = E \cap ((V_0 \cup O_p) \times (V_0 \cup O_p))$$

**Possible solution or hints:** This function terminates because on every recursion step, the parameter $n$ is decremented, and if $n = 0$, the function does not recurse.

**Explanation:** $C_p(0)$ is equal to $R_p$, the existing player routes for player $p$. On every recursion step, we take the routes that are reachable with one less vertex added, and extend them with one plausible edge.

We define the plausible edges in $E_{p,0}$ as being between any nodes that are in $V_0 \cup O_p$ (nodes that the player already owns or could buy). This helper definition is not strictly necessary but simplifies the presentation. We use $E \cap \ldots$ to ensure that $E_{p,0}$ only contains edges that actually exist on the game board.

To add edges from $E_{p,0}$, we either prepended or appended them to the routes that we could construct with $n-1$ purchases. (We here use the $\circ$ operator to combine the relations, as in class, but using set comprehension also works fine).

The function template was confusingly labelled $d = x \mapsto \{$ in the exam as printed. I only became aware (and notified students) of it with about 1.5h to go. It is not 100% clear to me what confusions this may have lead to.

- 2 Correct termination argument
  - 1 Decreases if greater 0
  - 1 Stops at 0
- 2 Base case $R_p$ (accept solution that yields $\emptyset$ if computation consistently computes $C_p \setminus R_p$ instead of $C_p$ as above, i.e., only the new edges enabled by new purchaeses)
- 2 Recursion
  - 1 Recurse on $n-1$
  - 1 Use recursion result as relation
- 3 Prepend and append paths to recursive result
  - 1 Append
  - 1 Prepend
  - 1 Union of prepended-only/appended-only
  - -1 Both appends and prepends (effectively -2 due to missing Union)
- 3 Utilise edges over $V_0$ and $E_P$
  - -2 not filtered by $E$
  - -1 cannot construct edges from $O_p$ to $V_0$, only between $V_0$ and $V_0$
  - -2 cannot construct edges from $V_0$ to $V_0$

## Question 5 (18 Points)

Let $f_i : A_i \to [0, 1]$ for $i \in \mathbb{N}$, where $[0, 1]$ and all other intervals in this question are over $\mathbb{R}$.

(a) (4 Points) Let $f_1 = x \mapsto (2 - x)^2$. What is $A_1$ so that $f_1$ is a bijection?

**Possible solution or hints:** $A_1 = [1, 2]$. (Making the interval smaller would make it non-surjective. Extending $A_1$ to be a subset of $[1, 3]$ would make it non-injective. Other extensions would make it map outside of $[0, 1]$.)

4 Depending on $A_1$:

- -1 maps outside of range
- -1 (additionally) if at least half of $A_1$ is mapped to outside of $f_1$'s range
- -2 not injective
- -2 not surjective

(b) (4 Points) Let $f_2 = x \mapsto x^2$. Give an $A_2$ such that $f_2$ is surjective but not injective, or explain why that is not possible.

**Possible solution or hints:** One possible solution is $A_2 = [-1, 1]$. There are other solutions, such as $[-1, 0] \cup \{0.3\}$ or $[-1, -0.5] \cup [0, 0.5]$; the trick is to ensure that $\{|x| : x \in A_2\} = [0, 1]\}$ (here, $|x|$ is the absolute value of $x$) while also ensuring that there exist $x_1, x_2 \in A_2$ such that $x_1 \neq x_2$ but $x_1^2 = x_2^2$ (the latter of which holds iff $|x_1| = |x_2|$).

4 Depending on $A_2$:

- -2 not surjective
- -2 injective after all
- -2 maps outside of range

(c) (6 Points) Let $g : [-2, 2] \to A_3$ such that $g : x \mapsto x^3$. Choose $A_3$ so that $g$ is surjective, then give a function $f_3$ such that $f_3 \circ g$ is injective. Is $f_3 \circ g$ bijective?

**Possible solution or hints:** $A_3 = [-8, 8]$, otherwise $g$ is either not surjective or maps outside of its range. Note that $g$ is also injective, so to make $f_3 \circ g$ injective, it suffices to choose an injective $f_3$. One option is the following:

$$f_3(x) = \frac{x + 8}{16}$$

This function is also surjective (since $f_3(-8) = 0$, $f_3(8) = 1$, and $f_3$ continuous and monotonic), which makes $f_3 \circ g$ surjective and hence bijective.

Another possible answer would be

$$f_{3'}(x) = \frac{8 - x}{1000}$$

though $f_{3'}$ is not surjective (and hence $f_{3'} \circ g$ would be neither surjective nor bijective). Either answer would give full credit, if correctly classified as bijective / not bijective.

4 $A_3$ and $f_3$

- -2 wrong $A_3$
- -1 (additionally): $A_3$ has a finite number of elements or is $[0, 1]$
- -2 $f_3$ maps outside of range
- -2 $f_3$ is not injective

2 Correct assessment of surjectivity / bijectivity

- -0.5 stated surjectivity but did not explicitly mention bijectivity

(d) (4 Points) Let $f_4 = x \mapsto x^2$ and $A_4 = \mathbb{R}$. What is $f_4\left[\left[\frac{1}{4}, \frac{1}{2}\right]\right]$ ?

**Possible solution or hints:** One valid answer is to say that the answer is undefined, since fixpoints are only defined on functions whose domain and range are equal (an earlier exam had suffered from the same mistake and included a suitable note).

Another valid answer is to (implicitly) assume that we are extending the domain of $f_4$ so that $f_4 : \mathbb{R} \to \mathbb{R}$, in which case

$$f_4\left[[\frac{1}{4}, \frac{1}{2}]\right] = (0, \frac{1}{4}]$$

Giving only $(0, \frac{1}{4}]$ (or equivalent notation, such as $]0, \frac{1}{4}]$ or $[0, \frac{1}{4}] \setminus \{0\}$) was sufficient for full credit.

4 depending on the interval:

- 4 includes $(0, \frac{1}{4}]$
  - -1 includes $0$
  - -2 does not include a fixed point (e.g., excludes $(0, \frac{1}{16})$)
  - -1 includes $(\frac{1}{4}, \frac{1}{2}]$
  - -2 includes negative numbers
  - -2 includes numbers greater than $\frac{1}{2}$

## Symbols and Notation

You may use any of the symbols and notation below in your own answers, in addition to any standard arithmetic notation and notation that we discussed in class. You may at any time introduce helper definitions.

| | |
|---|---|
| $\mathbb{N}$ | The natural numbers, starting at 0 |
| $\mathbb{Z}$ | The integers |
| $\mathbb{R}$ | The real numbers |
| $\mathcal{P}(A)$ | The power set of the set $A$ |
| $\#S, \lvert S \rvert$ | The cardinality of set $S$ |
| $R^{1}, f^{-1}$ | The inverse of a relation $R$ or a function $f$ |
| $R[X], f[X]$ | closure of a set $X$ under a relation $R$, a set of relations $R$, or a function $f$ |
| $R \circ S, f \circ g$ | of relations and functions: their composition |
| $R^+$ | The transitive closure of relation $R$ |
| $[a, b]$ | closed interval from $a$ to $b$ (including $\{a, b\}$) |
| $(a, b)$ | open interval from $a$ to $b$ (excluding $\{a, b\}$) |
| $(a, b], [a, b)$ | half-open intervals from $a$ to $b$ |
| $\lfloor x \rfloor$ | rounding down $x$ |
| $\sum S$ | sum of all elements of $S$ |
| $\prod S$ | product of all elements of $S$ |
| $\bigcup S$ | union of all elements of $S$ |
| $\bigcap S$ | intersection of all elements of $S$ |
| $\bigcup_{a \in S} E(a), \bigcap_{a \in S} E(a)$ | generalised union / intersection of all sets $E(a)$ for every $a \in S$ |

**Answers to common questions about grading:**

- When marking fields in a *table of choices*, each row is worth one point. Each incorrect answer counts as $-1$, while each omitted or ambiguous answer (multiple checkmarks) counts as $\pm 0$. A negative points total on a table counts as zero points total for that table.
- In any question, if subquestion $x$ refers to subquestion $y$ with $x \neq y$, then grading for subquestion $x$ assumes that you answered subquestion $y$ correctly, even if you did not. However, what the correct answer for subquestion $x$ is may depend on your answer to subquestion $y$.