EDAA75 Exam

EDAA75-2023 Exam #1

2023-06-03

Things you CAN use during the exam:

Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam:

Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, …

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favourable assumption about what you intended to write.

A sheet with common symbols and notations and with information about grading is attached at the end.

Possible solution or hints: This version includes a reference solution, marked like this paragraph. **Note:** the reference solution often offers additional explanations / proof sketches beyond what the question asked for, to help students who use it to study for future exams. Students were not required to explain their anwers unless the question explicity requested an explanation.

This version includes **grading guidelines**, marked like this paragraph.

Good luck!

Question:	1	2	3	4	5	6	**Sum**
Max Points:	16	12	18	14	18	22	100
Points Reached:							

Total points:	100 + lab bonus
Points required for **3**:	50
Points required for **4**:	67
Points required for **5**:	85

Grading Template

Notation

- 2 foo
  - 1 quux
  - 1 blah
- 2 bar

means that there are 4 points to distribute; 2 for "foo" and 2 for "bar". Of the 2 points for "foo", 1 is for "quux" and 1 is for "blah". Count incorrect answers as 0 (but see below).

Alternatives

- 2 Take one of the two routes
  - **Route A**:
    - 2 Did *foo*
  - **Route b**:
    - 2 Did *bar*

Some questions have alternatives. Here, first check which approach the student took, then use the appropriate scheme.

Capping negative points

- 2 foo
  - -1 Did X
  - -1 Did Y
  - -1 Did Z
- 2 bar
- -3 Did everything in log space

Some questions use negative points. Negative points are capped to 0 at their parent. So an answer that does "foo" but also did X, Y, and Z gets $2 - 3 = -1$ points, but this is capped at 0, so if "bar" is correct, the total is still 2. However, if the answer "did everything in log space", then the -3 apply across "foo" and "bar", so that "did X" + "did everything in log space" would give zero points for the question in total.

Cases that are not covered

Students are creative. While grading, it may turn out that something is unfair, or that a special case needs consideration. *Make a note somewhere if you deviate from the grading scheme.*

- For one-off cases, use your best judgment.
- For repeating patterns:
  - Look for a generalisable rule, write it down
  - **If you have few (ca. ≤ 20) earlier exams to revisit**: consider doing so
  - **Otherwise (incl, you decide not to revisit)**: determine the maximum delta that your change can make, and add it to your notes (e.g., "Q3.5: -0.5 to +2"). We can later use this information to revisit only those exams where your change has a chance of affecting the grade.

Ideally, we can use a shared document to track this information, but keep in mind that there must be no personally identifiable information in such documents, so tracking it on the CS git is likely safest.

If In Doubt, Be Generous

Question 1 (16 Points)

Let $K \subset \mathbb{N}$ such that $K = \{k_1, k_2\}$, with $1 < k_1 < k_2$. Let $I_K \supseteq K$ the minimal superset of K with the property that $a \in I_K$ and $k \in K \implies a + k \in I_K$.

(a) (4 Points) Define an *injective* function $f_K : \mathbb{N} \to I_K$:

$$f_K = n \mapsto (n+1) \cdot k_1$$

2 Maps to I_K (i.e., multiples of k_1 plus multiples of k_2 only, excluding 0)
2 Injective

(b) (3 Points) Is your function surjective / bijective? Explain.

Possible solution or hints: No. Counter-example: Assume $k_1 = 2$, $k_2 = 3$. Then there is no $n \in \mathbb{N}$ s.th. $f_K(n) = 3$, even though $k \in I_K$.

1 Correct answer (most likely "no")
2 Explanation, e.g., counterexample

(c) (6 Points) Define a relation R_K such that $I_K = R_K[K]$. Your definition of R must not be recursive or circular.

$$R_K = \{\langle a, c\rangle | a \in \mathbb{N}, b \in \{k_1, k_2\}, c = a + b\}$$

1 R_K contains pairs, e.g., $\langle a, b\rangle$
2 a is from sufficiently large set, e.g., $\mathbb{N}$ or I_K
1 b is sum of a with something
2 The elements added to b are from $\{k_1, k_2\}$

(d) (3 Points) Is your R_K reflexive? Is it symmetric? Is it transitive?

Possible solution or hints: No to all of the above.

1 reflexive (most likely **no**)
1 symmetric (most likely **no**)
1 transitive (most likely **no**)

Question 2 (12 Points)

Assume that a, b, and c are atomic propositions in Boolean propositional logic.

(a) (4 Points) Is the propositional Boolean logic formula $a \rightarrow ((\neg b) \vee c)$ equivalent to $(b \wedge a) \rightarrow c$? Justify your statement using the techniques that we studied in the course.

Possible solution or hints: Several approaches possible; ANY of the approaches below is sufficient:

Solution approach #1: Truth Tables

a	b	c	a → ((¬b) ∨ c)	(b ∧ a) → c
F	**F**	**F**	T	T
T	**F**	**F**	T	T
F	T	**F**	T	T
T	T	**F**	**F**	**F**
F	**F**	T	T	T
T	**F**	T	T	T
F	T	T	T	T
T	T	T	T	T

Solution approach #2: Transform one term to the other via tautological equivalence Example:

$$
\begin{aligned}
a \rightarrow ((\neg b) \vee c) &\iff a \rightarrow (b \rightarrow c) && (\rightarrow \text{intro}) \\
&\iff (a \wedge b) \rightarrow c && (\text{uncurrying}) \\
&\iff (b \wedge a) \rightarrow c && (\wedge\text{-commutativity})
\end{aligned}
$$

Solution approach #3: Transform both to DNF or other normal form, compare Both formulae transform into the following (first formula: by expanding definition of →; second formula: by expanding def. of → followed by applying DeMorgan equivalence):

$$(\neg a) \vee (\neg b) \vee c$$

1 Picked a suitable approach

3 Used the approach appropriately:

- **Approach 1**:
  - 2 Complete truth table (or equivalent summary)
    - · One "free" mistake at the level of a transcription error
    - -1 for misinterpreting implication semantics
    - -0.5 for individual mistakes not covered by the previous points
  - 1 Correct interpretation of results (even if table is incorrect)
- **Approaches 2/3**:
  - 2 Showed how derivation was done:
    - 1 Showed intermediate step (or final step, for #3)
    - 1 Showed additional intermediate step, or mentioned appropriate tautological equivalence for additional intermediate step
  - 0.5 Correct derivation
  - 0.5 Correct interpretation of results (even if derivation was wrong)

(b) (4 Points) Which of the following hold? For each row, mark if the statement **always** holds, **never** holds, or is **contingent**, in the choice table below:

Statement	always	never	contingent
$\vdash b \wedge \neg b$		X	
$\vdash a \vee (b \to \neg a)$	X		
$\neg b \vdash a \vee \neg(a \to b)$	X		
$\vdash (a \to b) \leftrightarrow (b \to \neg a)$		X	X
$b \vdash (a \leftrightarrow \neg c) \vee ((\neg(c \vee b)) \to a)$	X		

To undo a checkmark, circle it.

Note on grading: Each row (except for the example) is worth one point. Each incorrect answer counts as -1, while each omitted or ambiguous answer (multiple checkmarks) counts as ± 0. A negative points total on this table counts as zero points total for the table.

Possible solution or hints: The next-to-last statement is never true, since the empty left-hand side of the $\vdash$ (equivalent to $\models$ in SLAM, as discussed in class) specifies that we know nothing about a and b and that it would therefore have to hold for any truth assignment for these variables for the statement itself to hold.

However, this type of question was treated differently in earlier exams, so we also allow the answer "contingent".

- One point per question
- second-to last question: either answer is fine

(c) (4 Points) Are the following formulas ϕ and ψ equivalent for every S and every T? Explain your answer.

$$\begin{aligned} \phi &= \forall x \in S.\,(\exists y \in S.\,((x + y \in S) \vee (x \in T))) \\ \psi &= \exists y \in S.\,(\forall x \in S.\,((x + y \in S) \vee (x \in T))) \end{aligned}$$

Possible solution or hints: No. Counter-example: consider $S = \emptyset, S = \{-3, -1, 2, 3\}$: here, ϕ holds, since for $x \in \{-3, -1\}$, we can set $y = 2$, for $x = 2$, we can set $y = -3$, and for $x = 3$, we can set $y = -1$. However, ψ does not hold: there is no single y such that for all x, $x + y$ is in S.

1 Correct answer

3 Suitable explanation:

- 1 Answer shows understanding that $\forall\exists$ is not the same as $\exists\forall$
- 1 Answer shows understanding of *why* $\forall\exists$ is not the same as $\exists\forall$
- 1 Suitable argument (counter-example, high-level description)

Question 3 (18 Points)

Assume that $\langle T, R\rangle$ is a nonempty directed tree with $n = \#T$ elements and root a, and an injective function $f : D \to T$, where $D = [0, n)_{\mathbb{N}} = \{0, \ldots, n-1\}$ is an interval in $\mathbb{N}$. f has the following properties:

(P_1) $f(0) = a$

(P_2) For any $p, c \in D$, if $\langle f(p), f(c)\rangle \in R$, then $p < c$.

(P_3) $f^{-1} : T \to D$ is also an injective function.

(a) (6 Points) Let $x, y \in D$. Then $f(x)$ and $f(y)$ are tree nodes. If we know that $x = 2 \cdot y$, can there be a path from $f(x)$ to $f(y)$? Must there be such a path? Explain in your own words.

Possible solution or hints: There can be no such path. Proof by case distinction and contradiction:

- case $x = y = 0$: Such a path would be from a to a (due to (P_1), which contradicts the requirement that in directed trees, there must be no such path.
- case $x \geq 1$: This implies that $x > y$. By (P_2), any edge that goes to $f(y)$ has to come from a node index z such that $z < y$. Thus, for any path from x to y, all edges $\langle l, r\rangle$ on that path would have the property that $f^{-1}(l) < f^{-1}(r)$. By transitivity of $<$, such a path can only exist if $x < y$, which contradicts $x > y$.

2 case $x = 0$

- 0.5 Aware that this is special case
- 0.5 "not possible" in this case
- 1 Utilises tree axioms or other suitable properties from class
  - =0.5 partial credit if not utilised tree axioms but explicitly utilised (P_1)

4 case $x > 0$

- 1 "not possible" in this case
  - =0 Arrived at wrong conclusion because of reading $x = 2 \cdot y$ as $y = 2 \cdot x$ (can still get up to 3 of 4 points for this part of the question)
- 1 Utilises (P_2) in **explanation**
- 2 Utilises transitivity of $<$ (no need to name explicitly) in **explanation**, i.e. does not only look at parent/child, but also ancestor/descendant
- -0.5 Mixes up elements from D and from T in formal notation at least twice
- -0.5 Mixes up elements from D and from T in all cases

(b) (8 Points) Define a recursive function $d : D \to \mathbb{N}$ such that $d(x)$ is the *depth* of $f(x)$, i.e., the length of the path from the root a to $f(x)$, or 0 if there is no such path.

$$d = x \mapsto \begin{cases} 0 & \text{if } x = 0 \\ 1 + d(f^{-1}(R^{-1}(f(x)))) & \text{if } x > 0 \end{cases}$$

2 Case distinction 0 vs non-0
2 Recursive call on the parent index
  -1 Recursion uses parent node (from T) instead of index (from D)
2 Utilise R directly or indirectly to obtain parent
1 Use f^{-1} or equivalent
1 Use f or equivalent
-3 Recursion can never terminate (e.g., recursion on self– ignore distinction between D and T for this)
-3 Recursion can terminate, but is trivial (e.g., jumps directly to root)
-2 Recursion can terminate, but goes the wrong way (descendants, siblings, …)
-1 Recursion can terminate, but hops over parent (e.g., grandparent)

(c) (4 Points) Let $S = R \circ R^{-1}$. Which of the following properties hold for S? Mark exactly *one* field per row (**always**, **never**, **sometimes**) in the table below:

Property	**always**	**never**	**sometimes**
Reflexive		X	
Symmetric	X		
Antisymmetric			X
Transitive	X		

To undo a checkmark, circle it.

Note on grading: Each row is worth one point. Each incorrect answer counts as -1, while each omitted or ambiguous answer (multiple checkmarks) counts as ± 0. A negative points total on this table counts as zero points total for the table.

Possible solution or hints: Explanations: S represents a "sibling-or-self" relationship (except for a), i.e., "both nodes have a parent, and that parent is identical".

- S cannot be **reflexive**, because aSa never holds.
- S is always **symmetric** (to prove this formally, try showing e.g. that $S = S^{-1}$)
- S is **antisymmetric** for trees in which each node has at most one child, but not in trees in which any node has more than one child.
- S is always **transitive** (to prove this formally, try showing that aSb iff there exists a node p such that pRa and pRb and use the uniqueness of the parent for each node)

- One point per entry

Anonymkod:	Personal Code:

Question 4 (14 Points)

In a certain game, a set of players P are competing to *connect* different train stations via fields that the players can buy. We can think of the fields as vertices on a directed graph $\langle V, E\rangle$, with the train stations $S \subset V$, where the edges E are directed connections between two fields. Only one player $p \in P$ at a time may *own* a field, and we write $v \in O_p$ if and only if player p owns vertex v.

Analogously to *paths*, assume that a *route* from $v_a, v_b \in V$ is a sequence of vertices $v_0 \ldots v_k$ with $k \in \mathbb{N}$, where $v_0 = v_a$ and $v_k = v_b$ and $\langle v_i, v_{i+1}\rangle \in E$ for all $i \in \mathbb{N}, i < k$. For a given route, we call the set $\{v_i \mid i \in \mathbb{N}.i \leq k\}$ the *vertices on the route*.

A player p *is connecting* vertex $v_s \in V$ to vertex $v_e \in V$ if there is a route from v_s to v_e, and all vertices on that route are from $O_p \cup \{v_s, v_e\}$. Thus, a player can connect two train stations even if another player owns the stations themselves.

(a) (14 Points) Complete the definitions below:

- Define the set of *free vertices* $V_0 \subseteq V$ that are not owned by any player.

$$V_0 = V \setminus \bigcup_{p \in P} O_p$$

General: effective use of notation is key to this exercise. Interpret notational mistakes as wro answers if they introduce ambiguity.

3 Subtotal

1 Include all relevant vertices

2 Eliminate all player-owned vertices

- Define the relation $R_p \subseteq V \times V$ of all *player routes* for player p, such that $\langle v_1, v_2\rangle \in R_p$ if and only if there is a route from v_1 to v_2 where all vertices on the route are in O_p.

$$R_p = (E \cap (O_p \times O_p))^*$$

Possible solution or hints: Where S^* is the reflexive and transitive closure of the binary relation S. Alternatively, one can construct the transitive closure, $(E \cap (O_p \times O_p))^+$ and extend it with the identity relation $\{\langle v, v\rangle | v \in V\}$.

3 Subtotal

1 Uses transitive or reflexive+transitive closure

± 0 Ignore omission of reflexive closure, since it is reasonable in this application t only be interested in routes of lengths greater than zero.

1 Intersection that involves E

1 Work with $O_p \times O_p$

-1 No attempt to construct transitivity

- Define $T_p \subseteq S \times S$ such that $s_1 T_p s_2$ if and only if player p is connecting s_1 to s_2.

$$\begin{aligned} T_p &= (E \circ R_P \circ E) \cap (S \times S) \\ &= \{\langle s_1, s_2\rangle \mid s_1, s_2 \in S \wedge \exists v_1, v_2 \in V. \\ &\qquad (s_1 E v_1 \wedge v_1 R_p v_2 \wedge v_2 E s_2)\} \end{aligned}$$

Possible solution or hints: Either of the above definitions (or an equivalent definition) suffices.

4 Subtotal

1 Use R_p or equivalent
1 Prepend E or equivalent
1 Append E or equivalent
1 Draw from/filter by $S \times S$
-0.5 T_p includes connections that are not from $S \times S$

- Give a formula that holds if and only if a player is connecting all stations in the game to all other stations, in either direction (i.e., connecting s_1 to s_2 is sufficient, there is no need to also connect s_2 to s_1).

Possible solution or hints:

$$\exists p \in P.(\forall s_1, s_2 \in S.((s_1 \neq s_1) \rightarrow ((s_1 T_p s_2) \vee (s_2 T_p s_1))))$$

All of the parentheses in the above formula are optional; using standard precedence, we can write

$$\exists p \in P.\forall s_1, s_2 \in S.s_1 \neq s_1 \rightarrow s_1 T_p s_2 \vee s_2 T_p s_1$$

4 Subtotal

1 exists a player
1 check all nonequal pairs of stations
-0.5 Also checks paths from same station to itself
1 check for connecting in both directions
-0.5 require both instead of requiring either
1 use T_p or equivalent effectively

Question 5 (18 Points)

Let $f_i : A_i \to [0,1]$ for $i \in \mathbb{N}$, where $[0,1]$ and all other intervals in this question are over $\mathbb{R}$.

(a) (4 Points) Let $f_1 = x \mapsto (2-x)^2$. What is A_1 so that f_1 is a bijection?

Possible solution or hints: $A_1 = [1,2]$. (Making the interval smaller would make it non-surjective. Extending A_1 to be a subset of $[1,3]$ would make it non-injective. Other extensions would make it map outside of $[0,1]$.)

4 Depending on A_1:

- -1 maps outside of range
- -1 (additionally) if at least half of A_1 is mapped to outside of f_1's range
- -2 not injective
- -2 not surjective

(b) (4 Points) Let $f_2 = x \mapsto x^2$. Give an A_2 such that f_2 is surjective but not injective, or explain why that is not possible.

Possible solution or hints: One possible solution is $A_2 = [-1,1]$. There are other solutions, such as $[-1,0] \cup \{0.3\}$ or $[-1,-0.5] \cup [0,0.5]$; the trick is to ensure that $\{|x| : x \in A_2\} = [0,1]\}$ (here, $|x|$ is the absolute value of x) while also ensuring that there exist $x_1, x_2 \in A_2$ such that $x_1 \neq x_2$ but $x_1^2 = x_2^2$ (the latter of which holds iff $|x_1| = |x_2|$).

4 Depending on A_2:

- -2 not surjective
- -2 injective after all
- -2 maps outside of range

(c) (6 Points) Let $g : [-2,2] \to A_3$ such that $g : x \mapsto x^3$. Choose A_3 so that g is surjective, then give a function f_3 such that $f_3 \circ g$ is injective. Is $f_3 \circ g$ bijective?

Possible solution or hints: $A_3 = [-8,8]$, otherwise g is either not surjective or maps outside of its range. Note that g is also injective, so to make $f_3 \circ g$ injective, it suffices to choose an injective f_3. One option is the following:

$$f_3(x) = \frac{x+8}{16}$$

This function is also surjective (since $f_3(-8) = 0$, $f_3(8) = 1$, and f_3 continuous and monotonic), which makes $f_3 \circ g$ surjective and hence bijective.

Another possible answer would be

$$f_{3'}(x) = \frac{8-x}{1000}$$

though $f_{3'}$ is not surjective (and hence $f_{3'} \circ g$ would be neither surjective nor bijective). Either answer would give full credit, if correctly classified as bijective / not bijective.

4 A_3 and f_3

- -2 wrong A_3
- -1 (additionally): A_3 has a finite number of elements or is $[0,1]$
- -2 f_3 maps outside of range
- -2 f_3 is not injective

2 Correct assessment of surjectivity / bijectivity

- -0.5 stated surjectivity but did not explicitly mention bijectivity

(d) (4 Points) Let $f_4 = x \mapsto x^2$ and $A_4 = \mathbb{R}$. What is $f_4\left[\left[\frac{1}{4}, \frac{1}{2}\right]\right]$?

Possible solution or hints: One valid answer is to say that the answer is undefined, since fixpoints are only defined on functions whose domain and range are equal (an earlier exam had suffered from the same mistake and included a suitable note).

Another valid answer is to (implicitly) assume that we are extending the domain of f_4 so that $f_4 : \mathbb{R} \to \mathbb{R}$, in which case

$$f_4\left[[\frac{1}{4}, \frac{1}{2}]\right] = (0, \frac{1}{4}]$$

Giving only $(0, \frac{1}{4}]$ (or equivalent notation, such as $]0, \frac{1}{4}]$ or $[0, \frac{1}{4}] \setminus \{0\}$) was sufficient for full credit.

- 4 depending on the interval:
  - 4 includes $(0, \frac{1}{4}]$
    - -1 includes 0
    - -2 does not include a fixed point (e.g., excludes $(0, \frac{1}{16})$)
    - -1 includes $(\frac{1}{4}, \frac{1}{2}]$
    - -2 includes negative numbers
    - -2 includes numbers greater than $\frac{1}{2}$

Question 6 (22 Points)

In the game of Modified Mastermind, one player, the *codemaker*, chooses a *secret code*, a sequence of four colours from a set of six colours C, and the other player, the *codebreaker*, must guess this sequence by proposing a sequence of four colours to the codemaker for a limited number of rounds. Let $\mathbf{4} = \{0, 1, 2, 3\}$ the set of locations in the sequence; we can describe the secret code as $s : \mathbf{4} \to C$ and each guess g_i with $i \in \mathbb{N}$ as $g_i : \mathbf{4} \to C$.

After the codebreaker has made a guess g_i the codemaker reveals b_i, the number of locations for which the codebreaker guessed the correct colour, and w_i, the number of locations for which the codebreaker guessed the wrong colour but whose colour in the secret code matches the colour of another location that the codebreaker guessed incorrectly.

We formalise b_i and w_i as follows:

$$\begin{aligned} B_i &= \{\ell \mid \ell \in \mathbf{4}, g_i(\ell) = s(\ell)\} \\ b_i &= \#B_i \\ w_i &= \#\{\ell \mid \ell \in (\mathbf{4} \setminus B_i), s(\ell) \in g_i(\mathbf{4} \setminus B_i)\} \end{aligned}$$

Observe that $b_i + w_i \leq 4$.

The game exists in several variants:

- **Variant A**: The codemaker must not choose the same colour for different locations.
- **Variant B**: The codemaker may choose the same colour for different locations.
- **Variant C**: The codemaker must not choose the same colour for different locations, but may leave any number of locations empty. Otherwise, empty locations are treated as if they were an additional colour (i.e., in this variant, $\#C = 7$).

When answering the questions below, you do not need to evaluate exponentials, factorials, or products, sums, or quotients consisting of them.

(a) (4 Points) How many different secret codes can the codemaker choose from for **Variant A**?

Possible solution or hints: We are making an ordered choice among the *permutations*:

$$\frac{6!}{(6-4)!} = \frac{6!}{2} = 360$$

It was sufficient to provide the fraction.

1 Correct result, or solution implies or states that this is ordered choice
1 Correct result, or solution implies or states that this is permutation
2 Correct fraction or final result (or equivalent term)

(b) (4 Points) How many different secret codes can the codemaker choose from for **Variant B**?

Possible solution or hints: We are making an ordered choice among the *combinations*:

$$6^4 = 1296$$

It was sufficient to provide the exponential.

1 Correct result, or solution implies or states that this is ordered choice
1 Correct result, or solution implies or states that this is combination
2 Correct exponential, product, or final result (or equivalent term)

(c) (4 Points) How many different secret codes can the codemaker choose from for **Variant C**?

Possible solution or hints: This situation doesn't fit directly into one of our schemas, as it involves both permutations and combinations. We have to combine both, for all the possible number of holes (= empty locations) that there might be:

$$\sum_{n\in\{0,1,2,3,4} = \frac{6!}{(6-(4-n))!}\cdot\binom{4}{n}$$

Here, n represents the number of holes. $\frac{6!}{(6-(4-n))!}$ represents the different combinations, assuming $4-n$ ordered locations. We must then shuffle n holes among these locations, which means that we want unordered permutations: which give us $\binom{4}{n}$ is the number of different ways to distribute the n holes among the four locations.

While the above sum is a sufficient answer, we can expand it to:

$$\begin{aligned}
\textstyle\sum_{n\in\{0,1,2,3,4} = \frac{6!}{(6-(4-n))!}\cdot\binom{4}{n} &= \textstyle\sum_{n\in\{0,1,2,3,4} = \frac{6!}{(2+n))!}\cdot\binom{4}{n} \\
&= \tfrac{6!}{(2+0)!}\cdot\binom{4}{0}+\tfrac{6!}{(2+1)!}\cdot\binom{4}{1}+\tfrac{6!}{(2+2)!}\cdot\binom{4}{2}+\tfrac{6!}{(2+3)!}\cdot\binom{4}{3}+\tfrac{6!}{(2+4)!}\cdot\binom{4}{4} \\
&= \tfrac{6!}{2!}\cdot 1+\tfrac{6!}{3!}\cdot 4+\tfrac{6!}{4!}\cdot 6+\tfrac{6!}{5!}\cdot 4+\tfrac{6!}{6!}\cdot 1 \\
&= 360\cdot 1+120\cdot 4+30\cdot 6+6\cdot 4+1\cdot 1 \\
&= 360+480+180+24+1 \\
&= 1045
\end{aligned}$$

Note that the symmetry property of the binomial operator $\binom{n}{m} = \binom{n}{n-m}$ means that we also get the correct result if we write $\frac{6!}{(6-(4-n))!}\cdot\binom{4}{n}$ instead of $\frac{6!}{(6-n)!}\cdot\binom{4}{n}$; however, without an explanation, this is a mistake in the construction.

1 correct result, or identified need for summing over holes
1 correct result, or identified need for combinations
1 correct result, or identified need for permutations
1 correct result

(d) (5 Points) We are playing **Variant A**. The codebreaker has made a first guess g_1, with different colours for each location, and the codemaker has replied with $b_1 = 2$ and $w_1 = 0$. How many different secret codes remain that the codebreaker must now distinguish between? Show or explain how you arrived at your answer.

Possible solution or hints: The answer tells us two things:

(a) two locations are correct, but the codebreaker doesn't know which ones
(b) the two colours that the codebreaker did *not* pick must both occur in the answer

According to (a), two out of our four picks are completely correct. The number of possibilities for this case is the same as picking two out of four, ignoring order, i.e., $\binom{4}{2} = 6$ different possibilities.

For each of those possibilities, we must decide which of the two remaining colours (cf. (b)) we wish to place where — technically, this is a non-repeating, ordered choice, which means that we have $\frac{2!}{(2-2)!} = 2$ options. The total number of choices is then

$$\binom{4}{2}\cdot\frac{2!}{0!} = 6\cdot 2 = 12$$

1 Correctly identified (a)
1 Use (a) to identify $\binom{4}{2} = 6$ choices
1 Correctly identified (b)
1 Use (b) to identify $\frac{2!}{(2-2)!} = 2$ choices for the missing locations
1 Correct answer, or correct integration scheme for results from (a) and (b)

(e) (5 Points) We are playing **Variant B**. The codebreaker has made one guess, as follows:

$g_1(0)$	$g_1(1)$	$g_1(2)$	$g_1(3)$	w_1	b_1
c_0	c_1	c_1	c_2	1	0

where $c_0, c_1, c_2 \in C$ are three different colours. How many possible secret codes are there that would lead to this result? Show or explain how you arrived at your answer.

Possible solution or hints: Unfortunately, the question was ambiguously written. Below are answers to the two possible interpretations.

Interpretation #1: We know what c_1, c_2, and c_3 are The answer tells us that none of the colours in g_1 is in the right location, but that one of them occurs somewhere in s. With $w_1 = 1$, one of the following must hold:

- c_0 occurs exactly once in s at locations $\{1, 2, 3\}$, and c_1, c_2 do not occur, or
- c_1 occurs exactly once in s at locations $\{0, 3\}$, and c_0, c_2 do not occur, or
- c_2 occurs exactly once in s at locations $\{0, 1, 2\}$. and c_0, c_1 do not occur.

Note that these three options do not overlap, so we can compute the possible options for each case and then sum them up.

For each of the c_0 and c_2 cases, we have three locations for placing c_0/c_2, and for the remaining three locations we can choose three locations each, which gives:

$$3 \cdot 3^3$$

options for each of the two possibilities. For c_1, we analogously have

$$2 \cdot 3^3$$

which gives us

$$(3 + 3 + 2) \cdot 3^3 = 8 \cdot 3^3 = 8 \cdot 27 = 216$$

possibilities.

Interpretation #2: We do *not* know what c_1, c_2, and c_3 are An alternative interpretation of the question is that we do not know what c_0, c_1, and c_2 are. In that interpretation, we cannot simply compute the result from the result above by multiplying it with all possible choices for c_1, c_2, c_3, since these possibilities overlap — we would have to use the addition rule, which would be very complex to apply here.

Instead, first observe that knowing that two colours are not in s does not help us, since any given s will have at most 4 distinct colours anyway. The only information we can extract is that *one colour occurs exactly once in s*.

Describing the set of possible options directly is tedious, but we can instead consider all 6^4 possible unrestricted options and subtract those that are disabled by the restriction:

- All 6 patterns that consist of only one colour

- All $\binom{4}{2} = 6$ different patterns for splitting up four locations into two+two locations, times the unordered choice of $\binom{6}{2} = \frac{6!}{(6-2)!2!} = 15$ colour pairs to assign to these patterns.

This yields a total of

$$1296 - (6 + 15 * 6) = 1200$$

possibilities.

4 Depending on interpretation:

- **#1 (we know c_1, c_2, c_3):**
  - 1.5 identified the three cases (0.5 per case; discussion need to make it clear what they are but need not be as explicit as the above), incl **explanation**
  - 0.5 explicitly observed lack of overlap between the three cases (requires **explanation**)
  - 1 Correctly construct the 3^3 portion of the solution, **and explain**
  - 1 Correctly construct at least two of the $3/2/3$ possibilities for placing the result observed in w
    - -0.5 Only one of the possibilities is constructed correctly
    - -0.5 No explanation for where 3 and 2 come from
  - 1 Correct integration of the three partial results
- **#2 (we do not know c_1, c_2, c_3):**
  - 2 Correctly identify limited utility of our insights with explicit **explanation**
  - 1 Sketch of solution (sufficiently well **explained**)
  - 1 Apply suitable formulas
  - 1 Correct result

+1 Bonus: identified both possible readings

+2 Bonus: identified both possible readings AND one answer is correct or near-correct AND the other is sketched

Symbols and Notation

You may use any of the symbols and notation below in your own answers, in addition to any standard arithmetic notation and notation that we discussed in class. You may at any time introduce helper definitions.

$\mathbb{N}$	The natural numbers, starting at 0
$\mathbb{Z}$	The integers
$\mathbb{R}$	The real numbers
$\mathcal{P}(A)$	The power set of the set A
$\#S, \lvert S\rvert$	The cardinality of set S
R^{1}, f^{-1}	The inverse of a relation R or a function f
$R[X], f[X]$	closure of a set X under a relation R, a set of relations R, or a function f
$R \circ S, f \circ g$	of relations and functions: their composition
R^{+}	The transitive closure of relation R
$[a, b]$	closed interval from a to b (including $\{a, b\}$)
(a, b)	open interval from a to b (excluding $\{a, b\}$)
$(a, b], [a, b)$	half-open intervals from a to b
$\lfloor x \rfloor$	rounding down x
$\sum S$	sum of all elements of S
$\prod S$	product of all elements of S
$\bigcup S$	union of all elements of S
$\bigcap S$	intersection of all elements of S
$\bigcup_{a \in S} E(a), \bigcap_{a \in S} E(a)$	generalised union / intersection of all sets $E(a)$ for every $a \in S$
$\binom{n}{k}$	Binomial coefficients: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answers to common questions about grading:

- When marking fields in a *table of choices*, each row is worth one point. Each incorrect answer counts as -1, while each omitted or ambiguous answer (multiple checkmarks) counts as ± 0. A negative points total on a table counts as zero points total for that table.
- In any question, if subquestion x refers to subquestion y with $x \neq y$, then grading for subquestion x assumes that you answered subquestion y correctly, even if you did not. However, what the correct answer for subquestion x is may depend on your answer to subquestion y.