EDAA40 Exam

EDAA40-2023 Exam #2

2023-08-22

Things you CAN use during the exam:
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.
In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam:
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, …

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favourable assumption about what you intended to write.

A sheet with common symbols and notations and with information about grading is attached at the end.

Good luck!

Question:	1	2	3	4	5	6	**Sum**
Max Points:	16	16	16	20	14	18	100
Points Reached:							

Total points:	100 + lab bonus
Points required for **3**:	50
Points required for **4**:	67
Points required for **5**:	85

Question 1 (16 Points)

In this question, all intervals of the form $[a, b]$ are subsets of $\mathbb{Z}$.

If you encounter unfamiliar notation, make sure to check the notation table on the last page.

(a) Let $S = \{s \cdot n^2 \mid s \in \{-1, 1\}, n \in \mathbb{Z}\}$. Give a function $g : \mathbb{N} \to S$ such that g is injective, but not surjective.

$$g = x \mapsto$$

(b) Let $f_1 : [0, 1000] \to \mathbb{N}$ and $f_2 : \mathbb{N} \to [0, 1000]$. $f_2 = x \mapsto \lfloor \frac{x+1}{2} \rfloor$. Give f_1 such that $f_2 \circ f_1$ is injective.

$$f_1 = x \mapsto$$

(c) Consider the following function $h : \mathbb{N} \to \mathbb{N}$:

$$h(x) = \begin{cases} x - 1 & \text{if x is odd} \\ x + 2 & \text{if x is even} \end{cases}$$

What is $h([0, 1000])$? Answer without referring to h itself and without using a case split as in the definition of h.

$$h([0, 1000]) =$$

(d) Let $S = \{s \cdot n^2 \mid s \in \{-1, 1\}, n \in \mathbb{Z}\}$. Give a function $m : \mathbb{N} \to S$ such that m is bijective. (*Hint:* if you can think of a suitable m but don't know how to write it down, check the earlier questions for notational ideas.)

$$m = x \mapsto$$

Question 2 (16 Points)

Assume that a, b, and c are atomic propositions in Boolean propositional logic.

(a) Is the propositional Boolean logic formula $(\neg a) \vee \neg(b \wedge c)$ equivalent to $b \rightarrow (a \rightarrow \neg c)$? Justify your statement using one of the techniques that we studied in the course.

(b) Which of the following formulas hold? For each row, determine if the formula *always* holds, *never* holds, or is *contingent*. If the formula *always* holds, write *always* in the **Positive** column. If the formula *never* holds, write *never* in the **Positive** column. Otherwise give one example of a truth value assignment (e.g., $a = \mathbf{T}, b = \mathbf{F}$) that ensures that the formula holds (in the **Positive** column) and one example of a variable assignment that ensures that the formula does not hold (in the **Negative** column).

Formula	**Positive**	**Negative**
$b \wedge \neg b$	*never*	
$b \vee \neg b$	*always*	
a	$a = \mathbf{T}$	$a = \mathbf{F}$
$(a \vee b) \rightarrow \neg a$		
$a \vee \neg(b \rightarrow \neg a)$		
$\neg((\neg a) \wedge (b \vee \neg a))$		
$(a \leftrightarrow b) \wedge (a \leftrightarrow \neg b)$		

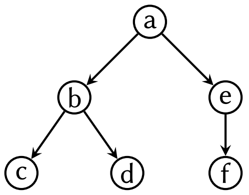
Note on grading: Each row (except for the three example rows) is worth two points, one per column. Omitting a column counts as ± 0. Answering a column incorrectly counts as -1. The answers *always*/*never* count for both columns. A negative points total on this table counts as zero points total for the table.

(c) Are the following formulas ϕ and ψ equivalent for every $S \subseteq \mathbb{Z}$? Explain your answer.

$$\phi = \forall x \in S. (\exists y \in S. (x \cdot y \in S))$$
$$\psi = \exists y \in S. (\forall x \in S. (x \cdot y \in S))$$

Question 3 (16 Points)

In this question we examine and characterise nodes that share ancestors. As an example, consider the following directed tree:



Siblings: {c, d}, {b, e}
First Cousins: {c, f}, {d, f}

Here, c and d are *siblings*, since they have the same parent, namely b. b and e are also siblings, with the same parent, a. Meanwhile, c and f are *first cousins*, since they share the same *grandparent* (parent-of-their-parent), namely a. Nodes d and f are also first cousins. However, a node is never its own sibling or its own first cousin, and two nodes that are siblings are not first cousins.

When we look at arbitrary directed trees (which may be of any size), we can generalise this idea to nth cousins: two nodes are *second cousins* when they share a great-grandparent, *third cousins* when they share a great-great-grandparent, and so on. In this format, a "0th cousin" is a sibling. As before, no node is its own nth cousin, and two nodes cannot be nth cousins if they are already mth cousins and $m < n$.

For an arbitrary (directed) tree $\langle T, R \rangle$, answer the following sub-questions. You may re-use definitions from *earlier* sub-questions (only), even if you have not solved those sub-questions.

(a) Define three relations $I \subseteq T \times T$, $S \subseteq T \times T$, and $Q \subseteq T \times T$, where xIy iff x and y are the same element, xSy whenever x and y have the same parent, and xQy whenever x and y have the same grandparent (= parent of their own parent).

$I =$

$S =$

$Q =$

(b) Now generalise S and Q. *Recursively* define a *family of relations* $P_n \subseteq T \times T$ with $n \in \mathbb{N}$ such that for all $n \geq 1$, the following holds: xP_ny iff x and y share some ancestor z, there exists a path from z to x of length n, and there exists a path from z to y of length n. In particular, $P_1 = S$ and $P_2 = Q$.
(Hint: you can choose P_0 freely, and it won't affect your grade. You may want to pick something that makes your life easy.)

$P_n =$

(c) Why is your recursive definition well-founded (i.e., why does it not loop infinitely)? Explain in your own words.

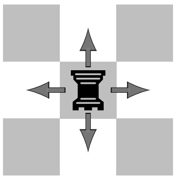
(d) Define another family of relations C_n with $n \in \mathbb{N}$ such that xC_ny iff x and y are nth cousins. (E.g., C_0 should describe the sibling relation.) Do *not* define C_n recursively. You may re-use any existing recursive definitions.

$$C_n =$$

Question 4 (20 Points)

In this question we develop a chess-like game that we call VChess, short for *variant chess.* You don't need to know anything about chess to answer the sub-questions below.

A VChess game involves different game pieces p that can move across the tiles of a game board. We can describe the game pieces' possible moves as vectors in two-dimensional Euclidean space. For example, the game piece below (♜) can move exactly one tile up, down, left, or right, and we can represent each of these movements as an element $\langle d_x, d_y \rangle \in \mathbb{Z}^2$, where d_x is the horizontal (left-right) movement and d_y is the vertical (up-down) movement:



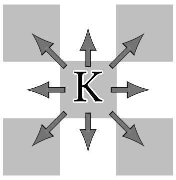
Move	d_x d_y
Left	$\langle -1, \ 0 \rangle$
Right	$\langle \ 1, \ 0 \rangle$
Down	$\langle \ 0, -1 \rangle$
Up	$\langle \ 0, \ 1 \rangle$

We call the set of all possible moves of some game piece p the *move set* of p, $M_p \subseteq \mathbb{Z}^2$. For example, a piece r that can move diagonally left-and-down has $\langle -1, -1 \rangle \in M_r$, and a piece q that can move arbitrarily far up has $\langle 0, n+1 \rangle \in M_q$ for all $n \in \mathbb{N}$. For our example above, we have:

$$M_{♜} = \{\langle -1, 0 \rangle, \langle 1, 0 \rangle, \langle 0, -1 \rangle, \langle 0, 1 \rangle\}$$

Solve the sub-questions below without giving any recursive definitions. Your definitions below may re-use your own definitions from earlier sub-questions (only).

(a) (Warm-up question) The game piece **K** can make the same moves as a ♜, or alternatively take one step in any of the four diagonal directions:



Give the move set of a **K**.

$$M_{\mathbf{K}} =$$

(b) Most game pieces' move sets follow a form of *rotational symmetry*: if the game piece can move to the right ($\langle 1, 0 \rangle \in M_p$), then the game piece can also move down ($\langle 0, -1 \rangle \in M_p$), or up, or to the left. Similarly, if the game piece can move diagonally up-and-left, it can also move diagonally in the other three directions.
Give a function $r_{90} : \mathbb{Z}^2 \to \mathbb{Z}^2$ that rotates a move by 90 degrees (clockwise or anticlockwise, your choice. You do *not* need trigonometric functions like $\sin$ and $\cos$ here.).

$$r_{90} =$$

(c) Describe this rotational symmetry by giving a relation R with the property that xRy iff x is rotationally symmetric to y, by which we mean that calling r_{90} repeatedly on x yields y (i.e., $y = r_{90}(r_{90}(\ldots r_{90}(x)\ldots)))$.

$$R =$$

(d) Construct a *rotational closure* function $c_R : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{P}(\mathbb{Z}^2)$ such that $c_R(M)$ is the smallest superset of M ($c_R(M) \supseteq M$) such that $m \in c_R(M)$ iff for all $m' \in \mathbb{Z}^2$ with mRm', we have that $m' \in c_R(M)$ holds.

$$c_R =$$

(e) Let $B \subseteq \mathbb{N}^2$ a *game board*, and all $t \in B$ the *tiles* in B. Further, let p a game piece with movement set M_p. We define that a move $m \in M_p$ *is valid for* p *at* $t \in B$ iff $m + t \in B$, where $m + t$ is defined as usual on vectors: If $m = \langle x_m, y_m \rangle$ and $t = \langle x_t, y_t \rangle$, then $m + t = \langle x_m + x_t, y_m + y_t \rangle$.

Informally, a *walk for game piece* p is a sequence of tiles that we can obtain as follows: first, we select an arbitrary tile for p that we call $f(0)$. Then, we make a valid move from $f(0)$ to $f(1)$, and keep making valid moves from $f(i)$ to $f(i+1)$ until p is on tile $f(n)$.

Let $n \in \mathbb{N}$ and $f : [0, n] \to B$ (where $[0, n] \subseteq \mathbb{N}$), Give a logical formula that holds if and only if f and n describe a walk for game piece p with move set M_p in the sense of the informal definition above.

Question 5 (14 Points)

Let $D_n = \{2 \cdot j + 3 \cdot k \mid j, k \in \mathbb{N}, n = j + k\}$, with $n \in \mathbb{N}$. Further, let $F \subset \mathbb{N} \times \mathbb{N}$ with $F = \{\langle n, m \rangle \mid n \in \mathbb{N}, m \in D_n\}$.

(a) What is the extension of the terms below?

$$D_0 =$$

$$D_1 =$$

$$D_2 =$$

(b) Is F transitive? Justify your answer.

s all possible x and y.

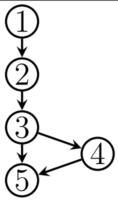
(c) Is F antisymmetric? Justify your answer.
(Hint: What is the smallest element in each D_n?)

Question 6 (18 Points)

In this question we analyse software with the help of graphs. A *guarded graph* is a tuple $\langle G, V, C, g, a\rangle$ where $\langle V, E\rangle$ is a graph of program statements V, with xEy stating that y can execute directly after x. Some edges $e \in E$ represent 'if' statements, which we model with *conditions* $C \supseteq \{\mathbf{T}\}$ and functions $g : E \to C$ and $a : V \to C$. We say that an edge $e \in E$ is *unguarded* iff $g(e) = \mathbf{T}$, and that e is *guarded* by $g(e)$ iff $g(e) \neq \mathbf{T}$. We say that vertex $v \in V$ *activates* condition c iff $a(v) = c$.

For a guarded graph $\langle V, E, C, g, a\rangle$, a *guarded path* of *length* $n-1$ from v_1 to v_n is a path $v_1, \ldots, v_n$ in $\langle V, E\rangle$, where additionally for all $j \in [1, n)$, $g(\langle v_j, v_{j+1}\rangle) = c$ implies that either $c = \mathbf{T}$, or there exists some $k \in \mathbb{N}, k \leq j$ such that $a(v_k) = c$.

As an example, the program below has $C = \{\mathsf{b}, \mathbf{T}\}$, and $\langle V, E\rangle$, g, and a as follows:

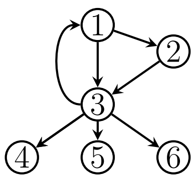
Program (for illustration only)	**Graph** $\langle V, E\rangle$	$g : E \to C$	$a : V \to C$
`start();`		$g(\langle 1,2\rangle) = \mathbf{T}$	$a(1) = \mathbf{T}$
`b = true;`		$g(\langle 2,3\rangle) = \mathbf{T}$	$a(2) = \mathsf{b}$
`if (b) {`		$g(\langle 3,4\rangle) = \mathsf{b}$	$a(3) = \mathbf{T}$
`    secret();`		$g(\langle 3,5\rangle) = \mathbf{T}$	$a(4) = \mathbf{T}$
`}`		$g(\langle 4,5\rangle) = \mathbf{T}$	$a(5) = \mathbf{T}$

Here, the sequence 1, 2, 3, 4, 5 is a guarded path because all adjacent vertices are edges, and because the only guarded edge, $\langle 3, 4\rangle$, is guarded by b, which vertex 2 activates (i.e., $a(2) = \mathsf{b}$).

However, $3, 4, 5$ is *not* a guarded path, because $\langle 3, 4\rangle$ is guarded by b, vertex 3 does not activate b, and there is no other vertex before 3 in this path that could activate b.

Answer the sub-questions below.

(a) Consider the following guarded graph with $C = \{\mathbf{T}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$:

Graph $\langle V, E\rangle$	$g : E \to C$	$a : V \to C$
	$g(\langle 1,2\rangle) = \mathsf{d}$	$a(1) = \mathbf{T}$
	$g(\langle 1,3\rangle) = \mathbf{T}$	$a(2) = \mathsf{c}$
	$g(\langle 2,3\rangle) = \mathbf{T}$	$a(3) = \mathsf{d}$
	$g(\langle 3,1\rangle) = \mathbf{T}$	$a(4) = \mathsf{e}$
	$g(\langle 3,4\rangle) = \mathsf{e}$	$a(5) = \mathbf{T}$
	$g(\langle 3,5\rangle) = \mathsf{c}$	$a(6) = \mathbf{T}$
	$g(\langle 3,6\rangle) = \mathbf{T}$	

List all vertices that you can reach with a guarded path of length n, starting at vertex 1:

Length	**Reachable vertices**
$n = 1$:	
$n = 2$:	
$n = 3$:	
$n = 4$:	
$n = 5$:	

(b) We analyse guarded graphs $\langle E, V, C, g, a\rangle$ with the help of *abstract program states*:
Let $S = \{\langle v, A\rangle \mid v \in V, \{\mathbf{T}, a(v)\} \subseteq A \subseteq C\}$. Then any $s \in S$ with $s = \langle v, A\rangle$ is an *abstract program state* at vertex v, and s *satisfies* condition $c \in C$ iff $c \in A$.

Define a *successor* relation $N \subseteq S \times S$ such that $s_1 N s_2$ if and only if s_1 is at vertex v_1 and s_2 is at vertex v_2, and there is an edge from v_1 to v_2 that is either unguarded or guarded by some condition c such that s_1 satisfies c.

$$N =$$

(c) Is N from sub-question (b) *always*, *never*, or *sometimes* a function? Explain your answer.

(d) Use N from sub-question (b) to give a logical formula or a term in set theory that holds if and only if there is a guarded path from $v_s \in V$ to $v_e \in V$ for some $\langle E, V, C, g, a\rangle$.

Symbols and Notation

You may use any of the symbols and notation below in your own answers, in addition to any standard arithmetic notation and notation that we discussed in class. You may at any time introduce helper definitions.

$\mathbb{Z}$	The integers
$\mathbb{R}$	The real numbers
$\mathbb{N}$	The natural numbers, starting at 0
$\mathcal{P}(A)$	The power set of the set A
$\#S, \lvert S \rvert$	The cardinality of set S
$\mathsf{dom}(R), \mathsf{dom}(f)$	The domain of a binary relation R or a function f
$\mathsf{range}(R), \mathsf{range}(f)$	The range of a binary relation R or a function f
R^{-1}, f^{-1}	The inverse of a relation R or a function f
$R \circ S, f \circ g$	The composition of relations and the composition of functions
R^+	The transitive closure of relation R
$R[X], f[X]$	closure of a set X under a relation R, a set of relations R, or a function f
$[a, b]$	closed interval from a to b (including $\{a, b\}$)
(a, b)	open interval from a to b (excluding $\{a, b\}$)
$(a, b], [a, b)$	half-open intervals from a to b
$\lfloor x \rfloor$	rounding down x
$\sum S$	sum of all elements of S
$\prod S$	product of all elements of S
$\bigcup S$	union of all elements of S
$\bigcap S$	intersection of all elements of S
$\bigcup_{a \in S} E(a), \bigcap_{a \in S} E(a)$	generalised union / intersection of all sets $E(a)$ for every $a \in S$

Answers to common questions about grading:

- In any question, if sub-question x refers to sub-question y with $x \neq y$, then grading for sub-question x assumes that you answered sub-question y correctly, even if you did not. However, what the correct answer for sub-question x is may depend on your answer to sub-question y.