EDAA40 Exam

EDAA40-2023 Exam #2

2023-08-22

Things you CAN use during the exam:
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like.
In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam:
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, …

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favourable assumption about what you intended to write.

A sheet with common symbols and notations and with information about grading is attached at the end.

Possible solution or hints: This version includes a reference solution, marked like this paragraph. **Note:** the reference solution often offers additional explanations / proof sketches beyond what the question asked for, to help students who use it to study for future exams. Students were not required to explain their answers unless the question explicitly requested an explanation.

Good luck!

Question:	1	2	3	4	5	**Sum**
Max Points:	16	16	20	14	18	84
Points Reached:						

Total points:	100 + lab bonus
Points required for **3**:	50
Points required for **4**:	67
Points required for **5**:	85

Question 1 (16 Points)

In this question, all intervals of the form $[a, b]$ are subsets of $\mathbb{Z}$.

If you encounter unfamiliar notation, make sure to check the notation table on the last page.

(a) Let $S = \{s \cdot n^2 \mid s \in \{-1, 1\}, n \in \mathbb{Z}\}$. Give a function $g : \mathbb{N} \to S$ such that g is injective, but not surjective.

$$g = x \mapsto x^2$$

(b) Which of the following formulas hold? For each row, determine if the formula *always* holds, *never* holds, or is *contingent*. If the formula *always* holds, write *always* in the **Positive** column. If the formula *never* holds, write *never* in the **Positive** column. Otherwise give one example of a truth value assignment (e.g., $a = \mathbf{T}, b = \mathbf{F}$) that ensures that the formula holds (in the **Positive** column) and one example of a variable assignment that ensures that the formula does not hold (in the **Negative** column).

Formula	**Positive**	**Negative**
$b \land \neg b$	*never*	
$b \lor \neg b$	*always*	
a	$a = \mathbf{T}$	$a = \mathbf{F}$
$(a \lor b) \to \neg a$	$a = \mathbf{F}, b = \mathbf{T}$	$a = \mathbf{T}$
$a \lor \neg(b \to \neg a)$	$a = \mathbf{T}$	$a = \mathbf{F}, b = \mathbf{F}, c = \mathbf{F}$
$\neg((\neg a) \land (b \lor \neg a))$	$a = \mathbf{T}, b = \mathbf{T}$	$a = \mathbf{F}, b = \mathbf{T}$
$(a \leftrightarrow b) \land (a \leftrightarrow \neg b)$	*never*	

Note on grading: Each row (except for the three example rows) is worth two points, one per column. Omitting a column counts as ± 0. Answering a column incorrectly counts as -1. The answers *always*/*never* count for both columns. A negative points total on this table counts as zero points total for the table.

(c) Are the following formulas ϕ and ψ equivalent for every $S \subseteq \mathbb{Z}$? Explain your answer.

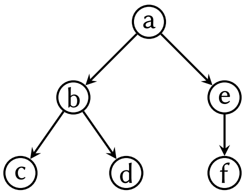
$$\begin{aligned} \phi &= \forall x \in S.\, (\exists y \in S.\, (x \cdot y \in S)) \\ \psi &= \exists y \in S.\, (\forall x \in S.\, (x \cdot y \in S)) \end{aligned}$$

Possible solution or hints: No. Counter-example: $S = \{-2, -1, 2\}$.

- For ϕ, $x \in \{-2, 2\}$ are covered by $y = -1$. With $x = -1$, we can select $y = 2$.
- For ψ, there is no single y that is suitable. $y = -1$ does not work with $x = -1$. $y \in \{-2, 2\}$ do not work with $x = 2$.

Question 2 (16 Points)

In this question we examine and characterise nodes that share ancestors. As an example, consider the following directed tree:



Siblings:	{c, d}, {b, e}
First Cousins:	{c, f}, {d, f}

Here, c and d are *siblings*, since they have the same parent, namely b. b and e are also siblings, with the same parent, a. Meanwhile, c and f are *first cousins*, since they share the same *grandparent* (parent-of-their-parent), namely a. Nodes d and f are also first cousins. However, a node is never its own sibling or its own first cousin, and two nodes that are siblings are not first cousins.

When we look at arbitrary directed trees (which may be of any size), we can generalise this idea to nth cousins: two nodes are *second cousins* when they share a great-grandparent, *third cousins* when they share a great-great-grandparent, and so on. In this format, a “0th cousin” is a sibling. As before, no node is its own nth cousin, and two nodes cannot be nth cousins if they are already mth cousins and $m < n$.

For an arbitrary (directed) tree $\langle T, R\rangle$, answer the following sub-questions. You may re-use definitions from *earlier* sub-questions (only), even if you have not solved those sub-questions.

(a) Define three relations $I \subseteq T \times T$, $S \subseteq T \times T$, and $Q \subseteq T \times T$, where xIy iff x and y are the same element, xSy whenever x and y have the same parent, and xQy whenever x and y have the same grandparent (= parent of their own parent).

$$I = \{\langle t, t\rangle \mid t \in T\}$$

$$S = R \circ R^{-1}$$

$$Q = R \circ R \circ R^{-1} \circ R^{-1}$$

(b) Now generalise S and Q. *Recursively* define a *family of relations* $P_n \subseteq T \times T$ with $n \in \mathbb{N}$ such that for all $n \geq 1$, the following holds: xP_ny iff x and y share some ancestor z, there exists a path from z to x of length n, and there exists a path from z to y of length n. In particular, $P_1 = S$ and $P_2 = Q$.
(Hint: you can choose P_0 freely, and it won’t affect your grade. You may want to pick something that makes your life easy.)

$$P_n = \begin{cases} I & \text{if } \; n = 0 \\ R \circ P_{n-1} \circ R^{-1} & \text{if } \; n > 0 \end{cases}$$

(c) Why is your recursive definition well-founded (i.e., why does it not loop infinitely)? Explain in your own words.

Possible solution or hints: Each P_{n+1} for $n \in \mathbb{N}$ references P_n, and P_0 does not reference any P_i for any $i \in \mathbb{N}$. Hence, each P_{n+1} will have precisely n recursion steps, as we can see by induction.

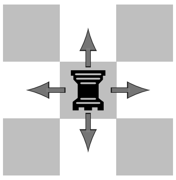
(d) Define another family of relations C_n with $n \in \mathbb{N}$ such that xC_ny iff x and y are nth cousins. (E.g., C_0 should describe the sibling relation.) Do *not* define C_n recursively. You may re-use any existing recursive definitions.

$$C_n = P_{n+1} \setminus P_n$$

Question 3 (20 Points)

In this question we develop a chess-like game that we call VChess, short for *variant chess*. You don't need to know anything about chess to answer the sub-questions below.

A VChess game involves different game pieces p that can move across the tiles of a game board. We can describe the game pieces' possible moves as vectors in two-dimensional Euclidean space. For example, the game piece below (♜) can move exactly one tile up, down, left, or right, and we can represent each of these movements as an element $\langle d_x, d_y \rangle \in \mathbb{Z}^2$, where d_x is the horizontal (left-right) movement and d_y is the vertical (up-down) movement:



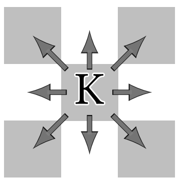
Move	d_x d_y
Left	$\langle -1, 0 \rangle$
Right	$\langle 1, 0 \rangle$
Down	$\langle 0, -1 \rangle$
Up	$\langle 0, 1 \rangle$

We call the set of all possible moves of some game piece p the *move set* of p, $M_p \subseteq \mathbb{Z}^2$. For example, a piece r that can move diagonally left-and-down has $\langle -1, -1 \rangle \in M_r$, and a piece q that can move arbitrarily far up has $\langle 0, n+1 \rangle \in M_q$ for all $n \in \mathbb{N}$. For our example above, we have:

$$M_{♜} = \{\langle -1, 0 \rangle, \langle 1, 0 \rangle, \langle 0, -1 \rangle, \langle 0, 1 \rangle\}$$

Solve the sub-questions below without giving any recursive definitions. Your definitions below may re-use your own definitions from earlier sub-questions (only).

(a) (Warm-up question) The game piece **K** can make the same moves as a ♜, or alternatively take one step in any of the four diagonal directions:



Give the move set of a **K**.

$$M_{\mathbf{K}} = M_{♜} \cup \{\langle 1, 1 \rangle, \langle -1, 1 \rangle, \langle 1, -1 \rangle, \langle -1, -1 \rangle\}$$

(b) Most game pieces' move sets follow a form of *rotational symmetry*: if the game piece can move to the right ($\langle 1, 0 \rangle \in M_p$), then the game piece can also move down ($\langle 0, -1 \rangle \in M_p$), or up, or to the left. Similarly, if the game piece can move diagonally up-and-left, it can also move diagonally in the other three directions.

Give a function $r_{90} : \mathbb{Z}^2 \to \mathbb{Z}^2$ that rotates a move by 90 degrees (clockwise or anticlockwise, your choice. You do *not* need trigonometric functions like $\sin$ and $\cos$ here.).

$$r_{90} = \langle d_x, d_y \rangle \mapsto \langle -d_y, d_x \rangle \text{ or } \langle d_y, -d_x \rangle$$

(c) Describe this rotational symmetry by giving a relation R with the property that xRy iff x is rotationally symmetric to y, by which we mean that calling r_{90} repeatedly on x yields y (i.e., $y = r_{90}(r_{90}(\ldots r_{90}(x)\ldots)))$.

$$R = \{\langle a, b\rangle \mid a \in \mathbb{Z}^2, b \in \mathbb{Z}^2, \\ b \in \{ a, \\ r_{90}(a), \\ (r_{90} \circ r_{90})(a), \\ (r_{90} \circ r_{90} \circ r_{90})(a)\} \quad \}$$

Possible solution or hints: Note that $r_{90} \circ r_{90} \circ r_{90} \circ r_{90}$ is the identity function $x \mapsto x$ (rotating by 360 degrees), so we don't need to consider rotating by 360 degrees, by 450 degrees etc.

(d) Construct a *rotational closure* function $c_R : \mathcal{P}(\mathbb{Z}^2) \to \mathcal{P}(\mathbb{Z}^2)$ such that $c_R(M)$ is the smallest superset of M ($c_R(M) \supseteq M$) such that $m \in c_R(M)$ iff for all $m' \in \mathbb{Z}^2$ with mRm', we have that $m' \in c_R(M)$ holds.

$$c_R = M \mapsto R(M)$$

(e) Let $B \subseteq \mathbb{N}^2$ a *game board*, and all $t \in B$ the *tiles* in B. Further, let p a game piece with movement set M_p. We define that a move $m \in M_p$ *is valid for* p *at* $t \in B$ iff $m + t \in B$, where $m + t$ is defined as usual on vectors: If $m = \langle x_m, y_m\rangle$ and $t = \langle x_t, y_t\rangle$, then $m + t = \langle x_m + x_t, y_m + y_t\rangle$.

Informally, a *walk for game piece* p is a sequence of tiles that we can obtain as follows: first, we select an arbitrary tile for p that we call $f(0)$. Then, we make a valid move from $f(0)$ to $f(1)$, and keep making valid moves from $f(i)$ to $f(i + 1)$ until p is on tile $f(n)$.

Let $n \in \mathbb{N}$ and $f : [0, n] \to B$ (where $[0, n] \subseteq \mathbb{N}$), Give a logical formula that holds if and only if f and n describe a walk for game piece p with move set M_p in the sense of the informal definition above.

Possible solution or hints:

$$\forall i \in [0, n-1].\exists m \in M_p.f(i) + m = f(i + 1)$$

Question 4 (14 Points)

Let $D_n = \{2 \cdot j + 3 \cdot k \mid j, k \in \mathbb{N}, n = j + k\}$, with $n \in \mathbb{N}$. Further, let $F \subset \mathbb{N} \times \mathbb{N}$ with $F = \{\langle n, m\rangle \mid n \in \mathbb{N}, m \in D_n\}$.

(a) What is the extension of the terms below?

$$D_0 = \{ 0 \}$$

$$D_1 = \{ 2, 3 \}$$

$$D_2 = \{ 4, 5, 6 \}$$

(b) Is F transitive? Justify your answer.

Possible solution or hints: No. $2 \in D_1$, so $1F2$, and $4 \in D_2$, so $2F4$. If F were transitive, then $1F4$ would have to hold, i.e., $4 \in D_1$, but that is not the case.

s all possible x and y.

(c) Is F antisymmetric? Justify your answer.
(Hint: What is the smallest element in each D_n?)

Possible solution or hints: Yes. To show that F is antisymmetric, we need to show that, for any $n, m \in \mathbb{N}$, if nFm, then either $\neg mFn$, or $n = m$.

The only $n, m \in \mathbb{N}$ with both nFm and mFn is $n = m = 0$, since $m \in D_n$ implies $m > n$ for all $n > 0$. (The smallest $x \in D_n$ is always $x = 2n$.)

To prove this formally (not required for full credit), we can use a proof by contradiction:

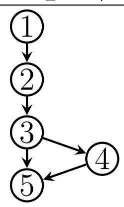
- Assume that there exist $n, m \in \mathbb{N}$ such that nFm and mFn with $n \neq m$.
- By definition of F, we know $m \in D_n$ and $n \in D_m$.
- Lemma 1: For any $x, y \in \mathbb{N}$, $x \geq x$ holds, and thus know that $x \in D_y$ implies that $x \geq \min(D_y)$, where $\min(D_y)$ is the smallest element in D_y.
- Lemma 2: For any $y \in \mathbb{N}$, $\min(D_y) = 2y$. Proof:
  - $2y \in D_y$ by construction (setting $j = y$ and $k = 0$).
  - Let $z \in \mathbb{N}$. If $z < 2y$, then $z \notin D_y$. Proof by contradiction: By construction, $z = 2j + 3k$ for $j + k = y, j, k \in \mathbb{N}$, i.e., $z = 2(y - k) + 3k = 2y + k$. Since $z \neq 2y$, we know $k > 0$, but then $z > 2y$, which contradicts $z < 2y$.
- From Lemma 1 and 2, we see that $m \geq 2n$, and $n \geq 2m$.
- Case split:
  - If (a) $m > n$, (b) $n > m$, or (c) both $n > m$ and $n > m$: With at least one of $m > n$ or $m > n$, it follows (transitively) that $n > n$, which is a contradiction.
  - Otherwise, $m = n$, which contradicts our premise.

Question 5 (18 Points)

In this question we analyse software with the help of graphs. A *guarded graph* is a tuple $\langle G, V, C, g, a\rangle$ where $\langle V, E\rangle$ is a graph of program statements V, with xEy stating that y can execute directly after x. Some edges $e \in E$ represent 'if' statements, which we model with *conditions* $C \supseteq \{\mathbf{T}\}$ and functions $g : E \to C$ and $a : V \to C$. We say that an edge $e \in E$ is *unguarded* iff $g(e) = \mathbf{T}$, and that e is *guarded* by $g(e)$ iff $g(e) \neq \mathbf{T}$. We say that vertex $v \in V$ *activates* condition c iff $a(v) = c$.

For a guarded graph $\langle V, E, C, g, a\rangle$, a *guarded path* of *length* $n-1$ from v_1 to v_n is a path $v_1, \ldots, v_n$ in $\langle V, E\rangle$, where additionally for all $j \in [1, n)$, $g(\langle v_j, v_{j+1}\rangle) = c$ implies that either $c = \mathbf{T}$, or there exists some $k \in \mathbb{N}, k \leq j$ such that $a(v_k) = c$.

As an example, the program below has $C = \{\mathsf{b}, \mathbf{T}\}$, and $\langle V, E\rangle$, g, and a as follows:

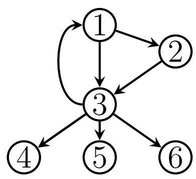
Program (for illustration only)	**Graph** $\langle V, E\rangle$	$g : E \to C$	$a : V \to C$
`start();`		$g(\langle 1,2\rangle) = \mathbf{T}$	$a(1) = \mathbf{T}$
`b = true;`		$g(\langle 2,3\rangle) = \mathbf{T}$	$a(2) = \mathsf{b}$
`if (b) {`		$g(\langle 3,4\rangle) = \mathsf{b}$	$a(3) = \mathbf{T}$
`  secret();`		$g(\langle 3,5\rangle) = \mathbf{T}$	$a(4) = \mathbf{T}$
`}`		$g(\langle 4,5\rangle) = \mathbf{T}$	$a(5) = \mathbf{T}$

Here, the sequence $1, 2, 3, 4, 5$ is a guarded path because all adjacent vertices are edges, and because the only guarded edge, $\langle 3, 4\rangle$, is guarded by b, which vertex 2 activates (i.e., $a(2) = \mathsf{b}$).

However, $3, 4, 5$ is *not* a guarded path, because $\langle 3, 4\rangle$ is guarded by b, vertex 3 does not activate b, and there is no other vertex before 3 in this path that could activate b.

Answer the sub-questions below.

(a) Consider the following guarded graph with $C = \{\mathbf{T}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$:

Graph $\langle V, E\rangle$	$g : E \to C$	$a : V \to C$
	$g(\langle 1,2\rangle) = \mathsf{d}$	$a(1) = \mathbf{T}$
	$g(\langle 1,3\rangle) = \mathbf{T}$	$a(2) = \mathsf{c}$
	$g(\langle 2,3\rangle) = \mathbf{T}$	$a(3) = \mathsf{d}$
	$g(\langle 3,1\rangle) = \mathbf{T}$	$a(4) = \mathsf{e}$
	$g(\langle 3,4\rangle) = \mathsf{e}$	$a(5) = \mathbf{T}$
	$g(\langle 3,5\rangle) = \mathsf{c}$	$a(6) = \mathbf{T}$
	$g(\langle 3,6\rangle) = \mathbf{T}$	

List all vertices that can be the last vertex of a guarded path that is of length n and starts at vertex 1.

[**NB**: in the exam as given, this was formulated as "List all vertices that you can reach with a guarded path of length n, starting at vertex 1"]

Length	**Reachable vertices**
$n = 1$:	{ 3 } (with d)
$n = 2$:	{6, 1 (with d)}
$n = 3$:	{2 (with c, d), 3 (with d)
$n = 4$:	{3 (with c, d), 6 (with d), 1 (with d)
$n = 5$:	{1,3 (with d), 2,5,6 (with c, d) }

Possible solution or hints: The above (minus the "with ... bits") is sufficient as a solution. Since "reachable" (in the version of the original exam) does not specifically only talk about the final node in the path, it was also correct to list all nodes traversed by the path, as long as the interpretation of "reachable" was consistent, possibly including 1. Under this alternative interpretation, the set of reachable vertices grows monotonically, with $n = 0$ implicitly being either $\emptyset$ or $\{1\}$. Then, $n = 1$ adds 3, $n = 2$ adds 6 and possibly 1, $n = 3$ adds 2, $n = 4$ does not add anything, and $n = 5$ adds 4 to the previous set.

(b) We analyse guarded graphs $\langle E, V, C, g, a\rangle$ with the help of *abstract program states*: Let $S = \{\langle v, A\rangle \mid v \in V, \{\mathbf{T}, a(v)\} \subseteq A \subseteq C\}$. Then any $s \in S$ with $s = \langle v, A\rangle$ is an *abstract program state* at vertex v, and s *satisfies* condition $c \in C$ iff $c \in A$.

Define a *successor* relation $N \subseteq S \times S$ such that $s_1 N s_2$ if and only if s_1 is at vertex v_1 and s_2 is at vertex v_2, and there is an edge from v_1 to v_2 that is either unguarded or guarded by some condition c such that s_1 satisfies c.

$$
\begin{aligned}
N = \{\ & \langle\langle v, A\rangle, \langle v', A'\rangle\rangle \in S \times S \\
\mid\ & \langle v, v'\rangle \in E \\
& \wedge g(\langle v, v'\rangle) \in A \\
& \wedge \{\mathbf{T}\} \subseteq A \\
& \wedge A' = A \cup \{a(v)\}\}
\end{aligned}
$$

(c) Is N from sub-question (b) *always*, *never*, or *sometimes* a function? Explain your answer.

Possible solution or hints: Sometimes: if each graph node has precisely one successor and all edges are unguarded, then there is exactly one successor vertex for every vertex and hence $N(s)$ exists and is is unique for any s. However, N is not a function in our earlier examples.

(d) Use N from sub-question (b) to give a logical formula or a term in set theory that holds if and only if there is a guarded path from $v_s \in V$ to $v_e \in V$ for some $\langle E, V, C, g, a\rangle$.

Possible solution or hints: $\exists A \subseteq C. \langle v_s, \{\mathbf{T}\}\rangle N^+ \langle v_e, A\rangle$, which excludes paths of length 0, as in class. Reflexive+transitive closure (N^*) was also acceptable for full credit.

Symbols and Notation

You may use any of the symbols and notation below in your own answers, in addition to any standard arithmetic notation and notation that we discussed in class. You may at any time introduce helper definitions.

$\mathbb{Z}$	The integers
$\mathbb{R}$	The real numbers
$\mathbb{N}$	The natural numbers, starting at 0
$\mathcal{P}(A)$	The power set of the set A
$\#S, \|S\|$	The cardinality of set S
$\mathsf{dom}(R), \mathsf{dom}(f)$	The domain of a binary relation R or a function f
$\mathsf{range}(R), \mathsf{range}(f)$	The range of a binary relation R or a function f
R^{-1}, f^{-1}	The inverse of a relation R or a function f
$R \circ S, f \circ g$	The composition of relations and the composition of functions
R^{+}	The transitive closure of relation R
$R[X], f[X]$	closure of a set X under a relation R, a set of relations R, or a function f
$[a, b]$	closed interval from a to b (including $\{a, b\}$)
(a, b)	open interval from a to b (excluding $\{a, b\}$)
$(a, b], [a, b)$	half-open intervals from a to b
$\lfloor x \rfloor$	rounding down x
$\sum S$	sum of all elements of S
$\prod S$	product of all elements of S
$\bigcup S$	union of all elements of S
$\bigcap S$	intersection of all elements of S
$\bigcup_{a \in S} E(a), \bigcap_{a \in S} E(a)$	generalised union / intersection of all sets $E(a)$ for every $a \in S$

Answers to common questions about grading:

- In any question, if sub-question x refers to sub-question y with $x \neq y$, then grading for sub-question x assumes that you answered sub-question y correctly, even if you did not. However, what the correct answer for sub-question x is may depend on your answer to sub-question y.