EDAA40 Exam

30 May 2016, 1400h-1900h

Instructions

Things you CAN use during the exam.
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like. Things I would recommend for this exam would be the book and the printed slides. If the full printout of the slides is a bit too much for your taste, use the handouts instead. They also have room for notes, so you can put in annotations that are useful to you.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam.
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favorable assumption about what you intended to write.

Note:

If you are asked to prove something, make sure you start by writing down the thing you need to prove. For example, if you are asked to show that some function f is surjective, you will usually need to show that the image of its domain is its codomain.

A sheet with common symbols and notations is attached at the end.

Good luck!

1 [6 p]

Given $A = \{a \in \mathbb{N}^+ : a \leq 6\}$, with $\mathbb{N}^+$ as always the natural numbers starting at 1, let us define the following sets (with $a|b$ iff $\exists k(k \in \mathbb{N}^+ \land ka = b)$):

$$B = \left\{ \frac{a}{b} : a, b \in A \right\}$$
$$C = \left\{ \frac{a}{b} : a, b \in A \land b|a \right\}$$
$$D = \left\{ \frac{a}{b} : a, b \in A \land a|b \right\}$$

Give the number of elements in these sets as follows:

1. [2 p] $\#(B) =$_______
2. [2 p] $\#(C) =$_______
3. [2 p] $\#(D) =$_______

Note that the question is about the *cardinality* of sets, so the answers are numbers.

2 [7 p]

Given $A = \{a, b, c, d, e, f\}$, what is

1. [1 p] $\#(\{s \in \mathcal{P}(A) : \#(s) = 0\}) =$_______
2. [1 p] $\#(\{s \in \mathcal{P}(A) : \#(s) = 1\}) =$_______
3. [1 p] $\#(\{s \in \mathcal{P}(A) : \#(s) = 2\}) =$_______
4. [1 p] $\#(\{s \in \mathcal{P}(A) : \#(s) = 3\}) =$_______
5. [1 p] $\#(\{s \in \mathcal{P}(A) : \#(s) = 4\}) =$_______
6. [1 p] $\#(\{s \in \mathcal{P}(A) : \#(s) = 5\}) =$_______
7. [1 p] $\#(\{s \in \mathcal{P}(A) : \#(s) = 6\}) =$_______

3 [8 p]

With $A = \{n \in \mathbb{N}^+ : n \leq 20\}$ and $R = \{(a, b) \in A^2 : a|b\}$ compute the following images of R:

1. [2 p] $R(6) =$
2. [2 p] $R(7) =$
3. [2 p] $R(2) =$
4. [2 p] $R(\{2, 5\}) =$

4 [19 p]

With $A = \{2, 3, 4, 5, 6, 7\}$, $R = \{(a, b) \in A^2 : a > b\}$, and $S = \{(a, b) \in A^2 : a|b\}$. We are looking at the composition $S \circ R$ in this task.

1. [3 p] $\#(S \circ R) =$_______
2. [3 p] $S \circ R(2) =$____
3. [3 p] $S \circ R(3) =$____
4. [3 p] $S \circ R(4) =$____
5. [3 p] $S \circ R(7) =$____
6. [4 p] $S \circ R$ is ... (circle those that apply)

reflexive symmetric transitive antisymmetric

5 [12 p]

Assume you have an injection $j : A \hookrightarrow B$ and a surjection $s : B \twoheadrightarrow C$.

1. [1 p] Is their composition $s \circ j$ always injective?

   YES NO

2. [5 p] If yes, prove that it is. If no, show a counterexample. (A counterexample involves making the three sets A, B, and C concrete, giving two functions for j and s with the required properties, and showing how their composition is not injective.)

3. [1 p] Is their composition $s \circ j$ always surjective?

   YES NO

4. [5 p] If yes, prove that it is. If no, show a counterexample. (A counterexample involves making the three sets A, B, and C concrete, giving two functions for j and s with the required properties, and showing how their composition is not surjective.)

6 [11 p]

Consider the lower-case alphabet $A = \{"a", ..., "z"\}$ and the set
$C = A \cup \{"(", ")", "\neg", " \vee ", " \wedge "\}$ of characters.
We define a small language $\mathcal{L} \subseteq C^*$ of propositional formulae over the set of variable names $V = A^* \setminus \{\varepsilon\}$, and the following set of rules $R = \{R_1, R_2, R_3\}$ with

$R_1 = \{(s, "\neg"s) : s \in C^*\}$
$R_2 = \{(s_1, s_2, "("s_1" \vee "s_2")") : s_1, s_2 \in C^*\}$
$R_3 = \{(s_1, s_2, "("s_1" \wedge "s_2")") : s_1, s_2 \in C^*\}$

such that $\mathcal{L} = R[V]$.

1. [1 p] Show that $\mathcal{L} \subset C^*$ by giving a string $s \in C^*$ such that $s \notin \mathcal{L}$:

   $s =$ ______________________________

   Hint: Make sure the strings are in $C^* \setminus \mathcal{L}$!

2. [3 p] Give three strings $s_1, s_2, s_3 \in C^* \setminus \mathcal{L}$ such that $(s_1, s_2, s_3) \in R_3$:

   $s_1 =$ ______________________________
   $s_2 =$ ______________________________
   $s_3 =$ ______________________________

3. [7 p] Assume a function $E : V \longrightarrow \{0, 1\}$ that assigns every variable name a value in $\{0, 1\}$. Using **structural recursion**, define an evaluation function $\mathrm{eval}_E : \mathcal{L} \longrightarrow \{0, 1\}$ that interprets the formulae in $\mathcal{L}$ in a way consistent with the usual interpretation of the symbols $\neg$, $\vee$, and $\wedge$ in propositional logic. **Use arithmetic operators (+, -, *, min, max) to compute with the values 0 and 1.**

$$\mathrm{eval}_E : s \mapsto \begin{cases} \underline{\qquad\qquad\qquad} & \text{for } s \in V \\ \underline{\qquad\qquad\qquad} & \text{for } s = \underline{\qquad\qquad} \\ \underline{\qquad\qquad\qquad} & \text{for } s = \underline{\qquad\qquad} \\ \underline{\qquad\qquad\qquad} & \text{for } s = \underline{\qquad\qquad} \end{cases}$$

7 [12 p]

Suppose we have a set A that is **totally ordered** by a relation $<$ on A. A function $f : A \longrightarrow A$ is called *strictly monotonic* iff for any $a, b \in A$ it is the case that $a < b \to f(a) < f(b)$.

1. [1 p] If a function f is strictly monotonic, does that imply it is **injective**? (circle answer)

   YES NO

2. [5 p] If yes, prove it. If no, provide a counterexample.
   (If you use a counterexample, you may use any totally ordered set that you find convenient, such as the natural/integer/rational/real numbers under the usual arithmetic order.)

3. [1 p] If a function f is strictly monotonic, does that imply it is **surjective**? (circle answer)

   YES NO

4. [5 p] If yes, prove it. If no, provide a counterexample.
   (If you use a counterexample, you may use any totally ordered set that you find convenient, such as the natural/integer/rational/real numbers under the usual arithmetic order.)

8 [21 p]

Let us use $\mathbb{P}$ as the name for the set of all prime numbers, that is positive integers greater than 1 that are only divisible by 1 and themselves, so $\mathbb{P} = \{2, 3, 5, 7, 11, 13, ...\}$. You can use $\mathbb{P}$ in answering the following questions, and also the “divides” relation, defined as $a|b$ iff $\exists k(k \in \mathbb{N}^+ \wedge ka = b)$.

1. [3 p] Give a definition of the set F_n of all prime factors of a positive natural number $n \in \mathbb{N}^+$, i.e. all prime numbers that are divisors of n.

   $F_n =$ ____________________________________

2. [6 p] The number *n primorial* is the product of all prime numbers less than or equal to n, i.e. $\prod\{p \in \mathbb{P} : p \leq n\}$. Let us call the function that computes n primorial P, so for example, $P(3) = 2 \cdot 3 = 6$, $P(4) = 2 \cdot 3 = 6$, $P(5) = 2 \cdot 3 \cdot 5 = 30$, $P(6) = 2 \cdot 3 \cdot 5 = 30$, $P(7) = 2 \cdot 3 \cdot 5 \cdot 7 = 210$ and so forth. The first primorial number is defined to be $P(1) = 1$.
   Using **simple recursion**, give a definition of the function $P : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ computing n primorial for any $n \in \mathbb{N}^+$, as follows:

$$P : n \mapsto \begin{cases} 1 & \text{for } n = 1 \\ \rule{3cm}{0.4pt} & \text{for } n > 1, n \notin \mathbb{P} \\ \rule{3cm}{0.4pt} & \text{for } n > 1, n \in \mathbb{P} \end{cases}$$

3. [2 p] Is the function $P : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$... (circle the answer)

(a)	injective?	YES	NO
(b)	surjective?	YES	NO

4. [4 p] Using **simple recursion**, define an **injective function** $Q : \mathbb{N}^+ \hookrightarrow \mathbb{P}$.
   Use the fact that for any $k \in \mathbb{N}^+$, the number $P(k) + 1$ is a prime number (a so-called *primorial prime*).
   Hint: It's NOT as simple as mapping n to $P(n) + 1$. (Make sure you understand why that is)

$$Q : n \mapsto \begin{cases} P(1) + 1 & \text{for } n = 1 \\ \underline{\hspace{6cm}} & \text{for } n > 1 \end{cases}$$

5. [6 p] Prove that Q above is injective.
   You may use the fact that $n \leq P(n)$ for all $n \in \mathbb{N}^+$ without needing to prove it.
   Hint: Answering this might become easier if you use a result from a previous task.

9 [7 p]

Let $A = \{a, b, c\}$ and $X = \{x, y\}$, and correspondingly A^* and X^* be the sets of finite sequences in A and X, respectively.

Using recursion over the structure of the sequence, define two **injections** $f : X^* \hookrightarrow A^*$ and $g : A^* \hookrightarrow X^*$, as follows. In both definitions, the first case deals with the empty sequence, the other cases “peel off” the first element in the sequence and the rest of the sequence is called s'.

$$f : s \mapsto \begin{cases} \underline{\qquad\qquad} & \text{for } s = \varepsilon \\ \underline{\qquad\qquad} & \text{for } s = xs', s' \in X^* \\ \underline{\qquad\qquad} & \text{for } s = ys', s' \in X^* \end{cases}$$

$$g : s \mapsto \begin{cases} \underline{\qquad\qquad} & \text{for } s = \varepsilon \\ \underline{\qquad\qquad} & \text{for } s = as', s' \in A^* \\ \underline{\qquad\qquad} & \text{for } s = bs', s' \in A^* \\ \underline{\qquad\qquad} & \text{for } s = cs', s' \in A^* \end{cases}$$

10 [4 p]

1. [2 p] Is the composition $f \circ g$ of the two functions defined in the previous task... (circle all that apply)

   injective surjective

2. [2 p] Is the composition $g \circ f$ of the two functions defined in the previous task... (circle all that apply)

   injective surjective

11 [9 p]

Identify free and bound occurrences of variables in the following formula. Put a dot **above** a free variable occurrence, and **below** a bound one.
Note that variable symbols immediately following quantifiers do not count as "occurrences".

free

$$Py \vee \exists z(Qxzy) \to \forall y(Rxy \leftrightarrow \exists y(Py \to \forall x(Rzx)))$$

bound

12 [15 p]

Find a DNF for each of the following formulae

1. [5 p] $\neg((r \vee q) \leftrightarrow (q \vee p))$

2. [5 p] $\neg((p \to q) \vee (q \to r) \vee (r \to p))$

3. [5 p] $(p \to q) \wedge (q \to r) \wedge (r \to p)$

Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e. including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k(k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- $\overline{R}$ of a relation R: its *complement*
- R^{-1} of a relation R: its *inverse*
- $R \circ S, f \circ g$ of relations and functions: their *composition*
- $R[X], f[X]$ *closure* of a set X under a relation R, a set of relations R, or a function f
- $[a, b],]a, b[,]a, b], [a, b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are *equinumerous*
- A^* for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, ε
- $\sum S$ sum of all elements of S
- $\prod S$ product of all elements of S
- $\bigcup S$ union of all elements in S
- $\bigcap S$ intersection of all elements in S