# EDAA40 Exam

26 August 2016, 0800h-1300h

## Instructions

**Things you CAN use during the exam.**
Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like. Things I would recommend for this exam would be the book and the printed slides. If the full printout of the slides is a bit too much for your taste, use the handouts instead. They also have room for notes, so you can put in annotations that are useful to you.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

**Things you CANNOT use during the exam.**
Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

**WRITE CLEARLY.** If I cannot read/decipher/make sense of something you write, I will make the least favorable assumption about what you intended to write.

## Note:

If you are asked to prove something, **make sure you start by writing down the thing you need to prove.** For example, if you are asked to show that some function $f$ is surjective, you will usually need to show that the image of its domain is its codomain.

**A sheet with common symbols and notations is attached at the end.**

Good luck!

## 1 [6 p]

Given $A = \{4, 5, 6, 7, 8, 9\}$, suppose we define the following sets (see the last page for the $\perp$ operator):

$$B = \left\{ \frac{a-b}{a+b} : a, b \in A \right\}$$

$$C = \left\{ \frac{a}{b} : a, b \in A \wedge a \perp b \right\}$$

Give the number of elements in these sets as follows:

1. [3 p] $\#(B) =$________

2. [3 p] $\#(C) =$________

Note that the question is about the *cardinality* of sets, so the answers are numbers.

## 2 [8 p]

With $A = \{n \in \mathbb{N}^+ : n \leq 20\}$ and $R = \{(a, b) \in A^2 : a \perp b\}$ compute the following images of R:

1. [2 p] $R(6) =$

2. [2 p] $R(7) =$

3. [2 p] $R(2) =$

4. [2 p] $R(\{2, 5\}) =$

# 3 [23 p]

With $A = \{2, 3, 4, 5, 6, 7\}$, $R = \{(a, b) \in A^2 : a \perp b\}$, and $S = \{(a, b) \in A^2 : a | b\}$. We are looking at the composition $S \circ R$ in this task.

1. [3 p] $\#(S \circ R) =$______
2. [3 p] $S \circ R(2) =$______________________________
3. [3 p] $S \circ R(3) =$______________________________
4. [3 p] $S \circ R(6) =$______________________________
5. [3 p] $S \circ R(7) =$______________________________
6. [4 p] $S \circ R$ is ... (circle those that apply)

| | | |
|---|---|---|
| … reflexive | TRUE | FALSE |
| … symmetric | TRUE | FALSE |
| … transitive | TRUE | FALSE |
| … antisymmetric | TRUE | FALSE |

7. [4 p] $R$ is ... (circle those that apply)

| | | |
|---|---|---|
| … reflexive | TRUE | FALSE |
| … symmetric | TRUE | FALSE |
| … transitive | TRUE | FALSE |
| … antisymmetric | TRUE | FALSE |

# 4 [12 p]

Assume you have a surjection $s : A \twoheadrightarrow B$ and an injection $j : B \hookrightarrow C$ .

1. [1 p] Is their composition $j \circ s$ always injective?

   YES NO

2. [5 p] If yes, prove that it is. If no, show a counterexample. (A counterexample involves making the three sets $A$, $B$, and $C$ concrete, giving two functions for $j$ and $s$ with the required properties, and showing how their composition is not injective.)

3. [1 p] Is their composition $j \circ s$ always surjective?

   YES NO

4. [5 p] If yes, prove that it is. If no, show a counterexample. (A counterexample involves making the three sets $A$, $B$, and $C$ concrete, giving two functions for $j$ and $s$ with the required properties, and showing how their composition is not surjective.)

# 5 [8 p]

As we saw in the lecture on quantificational logic, $\exists x \forall y Rxy$ always implies $\forall x \exists y Rxy$ for any relation *R*. The converse, however, is not necessarily the case: $\forall x \exists y Rxy$ does not always mean that $\exists x \forall y Rxy$ is true.

1. [6 p] Define a binary relation R over a non-empty universe D (that you also need to define) such that $\forall x \exists y Rxy$ is true, and $\exists x \forall y Rxy$ is false.
   Hint: Keep in mind that the $\forall$ and $\exists$ operators are quantified over D.

   D = ________________________________________

   R = ________________________________________

2. [2 p] Suppose $D = R = \emptyset$. What are the values of the formulae then?

| | | |
|---|---|---|
| $\forall x \exists y Rxy$ | TRUE | FALSE |
| $\exists x \forall y Rxy$ | TRUE | FALSE |

# 6 [17 p]

1. [2 p] Suppose we have a set $A$ partially ordered by a strict order $<$. What would you need to show in order to demonstrate that $(A, <)$ is **not** well-founded? (in English/Swedish)

2. [8 p] The set $\mathcal{P}(\mathbb{Q})$ under strict set inclusion $\subset$ is not well-founded. Show this.

3. [1 p] Is the set $\mathcal{P}(\mathbb{N})$ under strict set inclusion $\subset$ well-founded? (circle answer)

   YES NO

4. [6 p] Prove it or provide a counterexample. (Hint: You can use the bijection between $\mathbb{N}$ and $\mathbb{Q}$ we discussed in the course without having to define it here.)

# 7 [10 p]

Suppose we have a directed graph $(V, E)$. We want to define a function $r : V \longrightarrow \mathcal{P}(V)$ that computes for each vertex the set of vertices that one can reach from it by following zero or more directed edges.

We do this using a helper function $r' : V \times \mathcal{P}(V) \longrightarrow V$ that keeps track of the vertices we have visited already, so that we do not get stuck in cycles. Then $r$ itself simply becomes

$$r(a) = r'(a, \emptyset)$$

The second argument of $r'$ is the set of vertices we have visited already, initially empty.

Your task is to define $r'$ using recursion:

$$r' : (a, S) \mapsto \begin{cases} \rule{6cm}{0.4pt} & \text{for } a \in S \\ \rule{6cm}{0.4pt} & \text{otherwise} \end{cases}$$

Hint 1: Note that the edge relation $E$ can be used to compute all the nodes that can be reached from a given vertex $a$ in one hop: that set is simply the image of $a$ under $E$, i.e. $E(a)$.

Hint 2: You might want to recall the notion of a “generalized union”, along with the associated notation.

## 8 [9 p]

Identify free and bound occurrences of variables in the following formula. Put a dot **above** a free variable occurrence, and **below** a bound one.
Note that variable symbols immediately following quantifiers do not count as "occurrences".

free

$$(\exists y(Py \lor Qxzy)) \to \forall x(Rxy \leftrightarrow \exists y(Pz \to \forall x(Rzx)))$$

bound

## 9 [15 p]

Find a DNF for each of the following formulae. Write “none” if a formula has no DNF.

1. [5 p] $(r \lor q) \to (q \lor p))$

   ________________________________________

2. [5 p] $(p \to q) \to ((q \to r) \lor (r \to p))$

   ________________________________________

3. [5 p] $(p \to (q \land r)) \lor (q \to (p \land r)) \lor (r \to (p \land q))$

   ________________________________________

# Some common symbols

- $\mathbb{N}$ the natural numbers, starting at 0
- $\mathbb{N}^+$ the natural numbers, starting at 1
- $\mathbb{R}$ the real numbers
- $\mathbb{R}^+$ the non-negative real numbers, i.e. including 0
- $\mathbb{Z}$ the integers
- $\mathbb{Q}$ the rational numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k(k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- $\overline{R}$ of a relation R: its *complement*
- $R^{-1}$ of a relation R: its *inverse*
- $R \circ S, f \circ g$ of relations and functions: their *composition*
- $R[X], f[X]$ *closure* of a set X under a relation R, a set of relations R, or a function f
- $[a, b], ]a, b[, ]a, b], [a, b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are *equinumerous*
- $A^*$ for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, $\varepsilon$
- $\sum S$ sum of all elements of $S$
- $\prod S$ product of all elements of $S$
- $\bigcup S$ union of all elements in $S$
- $\bigcap S$ intersection of all elements in $S$
- $\bigcup_{a \in S} E(s)$ generalized union of the sets computed for every s in S