

EDAA40 Exam

26 August 2016, 0800h-1300h

Stuff in blue is part of the model solution.

Stuff in red is extra commentary.

Instructions

Things you CAN use during the exam.

Any written or printed material is fine. Textbook, other books, the printed slides, handwritten notes, whatever you like. Things I would recommend for this exam would be the book and the printed slides. If the full printout of the slides is a bit too much for your taste, use the handouts instead. They also have room for notes, so you can put in annotations that are useful to you.

In any case, it would be good to have a source for the relevant definitions, and also for notation, just in case you don't remember the precise definition of everything we discussed in the course.

Things you CANNOT use during the exam.

Anything electrical or electronic, any communication device: computers, calculators, mobile phones, toasters, ...

WRITE CLEARLY. If I cannot read/decipher/make sense of something you write, I will make the least favorable assumption about what you intended to write.

Note:

If you are asked to prove something, **make sure you start by writing down the thing you need to prove.** For example, if you are asked to show that some function f is surjective, you will usually need to show that the image of its domain is its codomain.

A sheet with common symbols and notations is attached at the end.

Good luck!

1**[6 p]**

Given $A = \{4, 5, 6, 7, 8, 9\}$, suppose we define the following sets (see the last page for the \perp operator):

$$B = \left\{ \frac{a-b}{a+b} : a, b \in A \right\}$$

$$C = \left\{ \frac{a}{b} : a, b \in A \wedge a \perp b \right\}$$

Give the number of elements in these sets as follows:

1. [3 p] $\#(B) = \underline{29}$

2. [3 p] $\#(C) = \underline{22}$

Note that the question is about the *cardinality* of sets, so the answers are numbers.

2**[8 p]**

With $A = \{n \in \mathbb{N}^+ : n \leq 20\}$ and $R = \{(a, b) \in A^2 : a \perp b\}$ compute the following images of R:

1. [2 p] $R(6) = \{1, 5, 7, 11, 13, 17, 19\}$

2. [2 p] $R(7) = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20\}$

3. [2 p] $R(2) = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

4. [2 p] $R(\{2, 5\}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$

3**[23 p]**

With $A = \{2, 3, 4, 5, 6, 7\}$, $R = \{(a, b) \in A^2 : a \perp b\}$, and $S = \{(a, b) \in A^2 : a|b\}$. We are looking at the composition $S \circ R$ in this task.

1. [3 p] $\#(S \circ R) = \underline{25}$
2. [3 p] $S \circ R(2) = \underline{\{3, 5, 6, 7\}}$
3. [3 p] $S \circ R(3) = \underline{\{2, 4, 5, 6, 7\}}$
4. [3 p] $S \circ R(6) = \underline{\{5, 7\}}$
5. [3 p] $S \circ R(7) = \underline{\{2, 3, 4, 5, 6\}}$
6. [4 p] $S \circ R$ is ... (circle those that apply)

... reflexive	TRUE	FALSE
... symmetric	TRUE	FALSE
... transitive	TRUE	FALSE
... antisymmetric	TRUE	FALSE

7. [4 p] R is ... (circle those that apply)

... reflexive	TRUE	FALSE
... symmetric	TRUE	FALSE
... transitive	TRUE	FALSE
... antisymmetric	TRUE	FALSE

4**[12 p]**

Assume you have a surjection $s : A \twoheadrightarrow B$ and an injection $j : B \hookrightarrow C$.

1. [1 p] Is their composition $j \circ s$ always injective?

YES

NO

2. [5 p] If yes, prove that it is. If no, show a counterexample. (A counterexample involves making the three sets A , B , and C concrete, giving two functions for j and s with the required properties, and showing how their composition is not injective.)

$$\begin{aligned} A &= C = \{a, c\}, B = \{b\} \\ s &= \{(a, b), (c, b)\}, j = \{(b, c)\} \\ j \circ s &= \{(a, c), (c, c)\} \end{aligned}$$

The composition is not injective, since $j \circ s(a) = j \circ s(c) = c$.

3. [1 p] Is their composition $j \circ s$ always surjective?

YES

NO

4. [5 p] If yes, prove that it is. If no, show a counterexample. (A counterexample involves making the three sets A , B , and C concrete, giving two functions for j and s with the required properties, and showing how their composition is not surjective.)

Same definitions as before. The composition is not surjective because $j \circ s(A) = \{c\} \neq C$.

One mistake that occurred a few times was giving definitions for j and s that were not, in fact, functions at all (either because they weren't unique mappings, or because they omitted values from the domain).

5**[8 p]**

As we saw in the lecture on quantificational logic, $\exists x \forall y Rxy$ always implies $\forall x \exists y Rxy$ for any relation R . The converse, however, is not necessarily the case: $\forall x \exists y Rxy$ does not always mean that $\exists x \forall y Rxy$ is true.

1. [6 p] Define a binary relation R over a non-empty universe D (that you also need to define) such that $\forall x \exists y Rxy$ is true, and $\exists x \forall y Rxy$ is false.

Hint: Keep in mind that the \forall and \exists operators are quantified over D .

$D =$ {a, b}

$R =$ {(a, a), (b, b)}

2. [2 p] Suppose $D = R = \emptyset$. What are the values of the formulae then?

$\forall x \exists y Rxy$ **TRUE** FALSE

$\exists x \forall y Rxy$ TRUE **FALSE**

Of course, there are many possible ways of constructing D and R . If they are empty, the first formula in the second part is true, because that is what happens when one universally quantifies over an empty set. Conversely, existentially quantifying over an empty set always yields false.

6

[17 p]

1. [2 p] Suppose we have a set A partially ordered by a strict order $<$. What would you need to show in order to demonstrate that $(A, <)$ is **not** well-founded? (in English/Swedish)

An infinite descending chain in A . [or]

A non-empty subset of A without a minimal element.

2. [8 p] The set $\mathcal{P}(\mathbb{Q})$ under strict set inclusion \subset is not well-founded. Show this.

$$S = \left\{ \left[0, \frac{1}{n} \right] : n \in \mathbb{N}^+ \right\}$$

S has no minimal element.

There are, of course, many ways of constructing an infinite descending chain. Including, incidentally, the one in the fourth part of this problem.

3. [1 p] Is the set $\mathcal{P}(\mathbb{N})$ under strict set inclusion \subset well-founded? (circle answer)

YES

NO

4. [6 p] Prove it or provide a counterexample. (Hint: You can use the bijection between \mathbb{N} and \mathbb{Q} we discussed in the course without having to define it here.)

$$S = \left\{ \{i : i \in \mathbb{N}^+, i > n\} : n \in \mathbb{N}^+ \right\}$$

S has no minimal element.

The bijection hint was meant for those who wouldn't find a simpler solution like the one above. If you found an answer to the second part of this problem, you already had an infinite descending chain in the rational numbers. With the bijection between those and the natural numbers, you could turn that into an infinite descending chain of sets of natural numbers "for free".

7**[10 p]**

Suppose we have a directed graph (V, E) . We want to define a function $r : V \rightarrow \mathcal{P}(V)$ that computes for each vertex the set of vertices that one can reach from it by following zero or more directed edges.

We do this using a helper function $r' : V \times \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ that keeps track of the vertices we have visited already, so that we do not get stuck in cycles. Then r itself simply becomes

$$r(a) = r'(a, \emptyset)$$

The second argument of r' is the set of vertices we have visited already, initially empty.

Your task is to define r' using recursion:

$$r' : (a, S) \mapsto \begin{cases} S & \text{for } a \in S \\ S \cup \{a\} \cup \bigcup_{v \in E(a)} r'(v, S \cup \{a\}) & \text{otherwise} \end{cases}$$

Hint 1: Note that the edge relation E can be used to compute all the nodes that can be reached from a given vertex a in one hop: that set is simply the image of a under E , i.e. $E(a)$.

Hint 2: You might want to recall the notion of a “generalized union”, along with the associated notation.

Judging from the responses, this seemed a very difficult problem.

The first case handles the case where we run into a node we have visited already, i.e. at the end of a cycle. In that case, we return all the nodes we have seen so far.

In the second case we deal with a node we haven't encountered before. We return the union of two things: $S \cup \{a\}$, the set of all nodes we have seen so far (plus the current one), and

$\bigcup_{v \in E(a)} r'(v, S \cup \{a\})$, the union of all those nodes we can reach from here along the edges going out from the current node (not forgetting to add the current node to the set of “visited nodes” we pass into r').

One might think that this second part would be sufficient, and that we would not need to explicitly add $S \cup \{a\}$ at the beginning, because all the calls to r' eventually run into the first case and will return the set S then. Except they don't: if a node has no outgoing edges, $E(a)$ is the empty set, and no calls to r' will be made. That is why we need to include $S \cup \{a\}$. Alternatively, one could write r' to explicitly distinguish the case where the current node has outgoing arcs from when it doesn't, which would add another case to the two given in the problem.

8 [9 p]

Identify free and bound occurrences of variables in the following formula. Put a dot **above** a free variable occurrence, and **below** a bound one.

Note that variable symbols immediately following quantifiers do not count as "occurrences".

free

$$(\exists y(Py \vee Qxzy)) \rightarrow \forall x(Rxy \leftrightarrow \exists y(Pz \rightarrow \forall x(Rzx)))$$

bound

9 [15 p]

Find a DNF for each of the following formulae. Write "none" if a formula has no DNF.

1. [5 p] $(r \vee q) \rightarrow (q \vee p)$

$$p \vee q \vee (\neg q \wedge \neg r)$$

2. [5 p] $(p \rightarrow q) \rightarrow ((q \rightarrow r) \vee (r \rightarrow p))$

$$p \vee \neg q \vee r \vee \neg r \vee (p \wedge \neg q) = p \vee \neg q \vee r \vee \neg r = \text{true}$$

3. [5 p] $(p \rightarrow (q \wedge r)) \vee (q \rightarrow (p \wedge r)) \vee (r \rightarrow (p \wedge q))$

$$\neg p \vee \neg q \vee \neg r \vee (p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$$

The last two DNFs evaluate to TRUE. That does not mean they don't exist, only that they aren't very interesting. For example, $p \vee \neg q \vee r \vee \neg r$ is a perfectly respectable DNF, even if it is obviously always TRUE.

Either way, responses that said that there was no DNF were scored incorrect. One exception was a response that said there was no DNF because the formulae were TRUE. I gave points for that one.

Also, any response that wasn't a DNF was marked incorrect.

Some common symbols

- \mathbb{N} the natural numbers, starting at 0
- \mathbb{N}^+ the natural numbers, starting at 1
- \mathbb{R} the real numbers
- \mathbb{R}^+ the non-negative real numbers, i.e. including 0
- \mathbb{Z} the integers
- \mathbb{Q} the rational numbers
- $a \perp b$ a and b are coprime, i.e. they do not have a common divisor other than 1
- $a \mid b$ a divides b, i.e. $\exists k(k \in \mathbb{N} \wedge ka = b)$
- $\mathcal{P}(A)$ *power set* of A
- \overline{R} of a relation R: its *complement*
- R^{-1} of a relation R: its *inverse*
- $R \circ S, f \circ g$ of relations and functions: their *composition*
- $R[X], f[X]$ *closure* of a set X under a relation R, a set of relations R, or a function f
- $[a, b],]a, b[,]a, b], [a, b[$ closed, open, and half-open intervals from a to b
- $A \sim B$ two sets A and B are *equinumerous*
- A^* for a finite set A, the set of all finite sequences of elements of A, including the empty sequence, ε
- $\sum S$ sum of all elements of S
- $\prod S$ product of all elements of S
- $\bigcup S$ union of all elements in S
- $\bigcap S$ intersection of all elements in S
- $\bigcup_{a \in S} E(s)$ generalized union of the sets computed for every s in S